Distributional Behavior of Convolution Sum System Representations

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***Abstract*—In this paper we study the validity of the usual convolution sum sampling representation of linear time-invariant (LTI) systems. We consider continuous input signals with finite energy that are absolutely integrable and vanish at infinity. Even for these benign signals, the convolution sum does not always converge. There exist LTI systems and signals such that the convolution sum diverges even in a distributional sense. This result shows that the practice of multiplying a signal with a Dirac comb and convolving subsequently with the impulse response of the LTI system is not valid for this signal space. We further fully characterize the LTI systems for which we have convergence for all signals in the space, and establish a connection between pointwise, uniform, and distributional convergence. In particular, we show that the convolution sum converges in a distributional sense if and only it converges in a classical pointwise sense. Hence, for this signal space, nothing can be gained by treating the convergence in a distributional sense.**

***Index Terms*—Linear time-invariant system, convolution sum, tempered distribution, divergence**

I. Introduction

Linear time-invariant (LTI) systems are often used in signal processing applications [1]–[4]. For bandlimited input signals f with finite energy and stable LTI systems T, the system output Tf can be computed using the frequency domain representation of the LTI system

$$(Tf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega)\hat{h}_T(\omega)\,\mathrm{e}^{i\omega t}\;\mathrm{d}\omega, \quad t \in \mathbb{R},$$

where $h_T = T(\mathrm{sinc})$ denotes the response of the system T to the sinc function. When dealing with bandlimited signals we can assume, without loss of generality, that the bandwidth is π, because any signal with a bandwidth other than π can be scaled to have this bandwidth. Another representation of the stable LTI system is given by the convolution sum sampling representation

$$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(k)h_T(t-k), \quad t \in \mathbb{R}, \tag{1}$$

which needs only the samples $\{f(k)\}_{k\in\mathbb{Z}}$ of the input signal f. For bandlimited signals f with finite energy, this series converges globally uniformly. However, sometimes the representation (1) is used in a much wider generality for other signal spaces. Then, it is known that the convergence of (1) can be problematic [5]–[8].

In this paper we study the convergence behavior of (1) for signals f in $\mathcal{C} = C_0(\mathbb{R}) \cap L^1(\mathbb{R})$. The precise definitions of $C_0(\mathbb{R})$ and $L^1(\mathbb{R})$ will follow in the next section. We show that the series in (1) does not necessarily converge, even if the convergence is treated in a distributional setting. More precisely, there exist stable LTI systems T such that (1) diverges for certain signals $f \in \mathcal{C}$. However, if, for a given stable LTI system, the series (1) converges for all $f \in \mathcal{C}$, then it converges even globally uniformly for all $f \in \mathcal{C}$.

The outline of the paper is as follows. After fixing notations and introducing distributions in Sections II and III, respectively, we give a motivation in Section IV. In Section V we present a necessary and sufficient condition for pointwise and uniform convergence. Then, in Section VII, we show that the same condition is sufficient and necessary for convergence in a distributional setting. Finally, we discuss the size of the set of systems and signals for which we have divergence in Section VI for classical divergence and in Section VIII for distributional divergence.

II. Notation

By c_0 we denote the set of all sequences that vanish at infinity. For $\Omega \subseteq \mathbb{R}$, let $L^p(\Omega)$, $1 \le p < \infty$, be the space of all measurable, pth-power Lebesgue integrable functions on Ω, with the usual norm $\|\cdot\|_p$, and $L^\infty(\Omega)$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. $C(\Omega)$, equipped with the supremum norm, is the space of continuous functions on Ω. By $C_0(\mathbb{R})$ we denote the Banach space of all continuous functions on $\mathbb{R}$ that vanish at infinity. The norm of this space is given by $\|f\|_{C_0(\mathbb{R})} = \max_{t\in\mathbb{R}}|f(t)|$.

Let $\mathcal{C} = C_0(\mathbb{R}) \cap L^1(\mathbb{R})$. Equipped with the norm $\|f\|_{\mathcal{C}} = \max\{\|f\|_{C_0(\mathbb{R})}, \|f\|_{L^1(\mathbb{R})}\}$, $\mathcal{C}$ becomes a separable Banach space. For all $f \in \mathcal{C}$ we have $\int_{-\infty}^{\infty}|f(t)|^2 \; \mathrm{d}t \le \|f\|_{C_0(\mathbb{R})} \int_{-\infty}^{\infty}|f(t)| \; \mathrm{d}t < \infty$, which shows that $f \in L^2(\mathbb{R})$, and consequently $\mathcal{C} \subset L^2(\mathbb{R})$. That is, all $f \in \mathcal{C}$ have finite energy. Let $\hat{f} = \mathcal{F}f$ denote the Fourier transform of a function f, where $\hat{f}$ is to be understood in the distributional sense. According to the Riemann–Lebesgue lemma, we have $\lim_{|t|\to\infty}|\hat{f}(t)| = 0$, i.e., $\hat{f} \in C_0(\mathbb{R})$, for all $f \in \mathcal{C}$, which shows that the Fourier transform of functions in $\mathcal{C}$ has nice properties.

The Bernstein space $\mathcal{B}_\sigma^p$, $\sigma > 0$, $1 \le p \le \infty$, consists of all functions of exponential type at most σ, whose restriction to the real line is in $L^p(\mathbb{R})$ [9, p. 49]. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line. A function in $\mathcal{B}_\sigma^p$

Parts of this work were presented at the Technische Universität Dresden in September 2017 at a ceremony celebrating the birthdays of Prof. Schreiber and Prof. Schwarz.

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is called bandlimited to σ. By $\mathcal{PW}_\sigma^p$, $1 \leq p \leq \infty$, we denote the Paley–Wiener space of functions f with a representation $f(z) = 1/(2\pi) \int_{-\sigma}^{\sigma} g(\omega) \,\mathrm{e}^{iz\omega} \,\mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma, \sigma]$. If $f \in \mathcal{PW}_\sigma^p$ then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\sigma^p$ is given by $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi) \int_{-\sigma}^{\sigma} |\hat{f}(\omega)|^p \,\mathrm{d}\omega)^{1/p}$. $\mathcal{PW}_\sigma^2$ is the frequently used space of bandlimited functions with finite energy.

III. Distributions

Distributions are continuous linear functionals on a space of test functions. Two common test functions spaces are $\mathcal{D}$ and $\mathcal{S}$. $\mathcal{D}$ is the space of all functions $\phi : \mathbb{R} \to \mathbb{C}$ that have continuous derivatives of all orders and are zero outside some finite interval. $\mathcal{D}'$ denotes the dual space of $\mathcal{D}$, i.e., the space of all distributions that can be defined on $\mathcal{D}$. The Schwartz space $\mathcal{S}$ consists of all continuous functions $\phi : \mathbb{R} \to \mathbb{C}$ that have continuous derivatives of all orders and fulfill $\sup_{t\in\mathbb{R}} |t^a \phi^{(b)}(t)| < \infty$ for all $a, b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $\mathcal{S}'$ denotes the dual space of $\mathcal{S}$. In this paper we will consider the convergence in $\mathcal{S}'$. A nice property of $\mathcal{S}$ is that the Fourier transform maps $\mathcal{S}$ onto itself, i.e., we have $\mathcal{F}\mathcal{S} = \mathcal{S}$. Clearly, we have $\|\phi\|_\infty < \infty$ and $\|\phi\|_1 < \infty$ for all $\phi \in \mathcal{S}$. In contrast to $\mathcal{S}$, the Fourier transform of functions in $\mathcal{D}$ is not necessarily again in $\mathcal{D}$. Hence, it is problematic to define the Fourier transform for $\mathcal{D}'$.

Typical examples of distributions in $\mathcal{S}'$ are the Dirac delta function δ, the derivatives of the Dirac delta function $\delta^{(l)}$, $l \geq 1$, all finite linear combinations of distributions, and the Dirac comb $Ш(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$.

For a locally integrable functions g we can define the linear functional

$$\phi \mapsto \int_{-\infty}^{\infty} g(t)\phi(t) \,\mathrm{d}t \tag{2}$$

on the space $\mathcal{D}$. It can be proved that this functional is continuous and thus defines a distribution [10]. If g further fulfills $\int_{-\infty}^{\infty} |g(t)|(1+|t|)^{-m} \,\mathrm{d}t < \infty$ for some $m \geq 0$ then (2) defines also a continuous linear functional on $\mathcal{S}$. Distributions of the type (2) are called regular distributions.

A sequence of distributions $\{f_k\}_{k\in\mathbb{N}}$ in $\mathcal{S}'$ is said to converge in $\mathcal{S}'$ if for every $\phi \in \mathcal{S}$ the sequence of numbers $\{f_k\phi\}_{k\in\mathbb{N}}$ converges. Thus, a sequence of regular distributions, which is induced by a sequence of functions $\{g_k\}_{k\in\mathbb{N}}$ according to (2), converges in $\mathcal{S}'$ if for every $\phi \in \mathcal{S}$ the sequence of numbers $\{\int_{-\infty}^{\infty} g_k(t)\phi(t) \,\mathrm{d}t\}_{k\in\mathbb{N}}$ converges. For further details about distributions, and for a definition of convergence in the test spaces, we would like to refer the reader to [10].

IV. Motivation

Before we start motivating our investigations, we review some facts about linear time-invariant (LTI) systems. A linear system T called time-invariant if $(Tf(\,\cdot - a))(t) = (Tf)(t-a)$ for all input signals f and all $t, a \in \mathbb{R}$. For each $h_T \in \mathcal{PW}_\pi^\infty$ the convolution integral

$$(Tf)(t) = \int_{-\infty}^{\infty} f(\tau) h_T(t-\tau) \,\mathrm{d}\tau, \quad t \in \mathbb{R}, \tag{3}$$

defines a LTI system on the space $\mathcal{C}$. Since $\mathcal{C} \subset L^1(\mathbb{R})$, this integral is even absolutely convergent for all $f \in \mathcal{C}$. Moreover, since $\mathcal{C} \subset L^2(\mathbb{R})$ and $\mathcal{PW}_\pi^\infty \subset \mathcal{PW}_\pi^2$, we can use the representation

$$(Tf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{h}_T(\omega) \,\mathrm{e}^{i\omega t} \,\mathrm{d}\omega, \quad t \in \mathbb{R}, \tag{4}$$

and Plancherel's theorem to obtain

$$\begin{aligned} \|Tf\|_{\mathcal{PW}_\pi^2}^2 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} |\hat{f}(\omega)|^2 |\hat{h}_T(\omega)|^2 \,\mathrm{d}\omega \\ &\leq \|h_T\|_{\mathcal{PW}_\pi^\infty}^2 \|\hat{f}\|_{L^\infty[-\pi,\pi]}^2 \\ &\leq \|h_T\|_{\mathcal{PW}_\pi^\infty}^2 \|f\|_1^2 \\ &\leq \|h_T\|_{\mathcal{PW}_\pi^\infty}^2 \|f\|_{\mathcal{C}}^2 \end{aligned}$$

for all $f \in \mathcal{C}$ and all $h_T \in \mathcal{PW}_\pi^\infty$. This shows that for all $h_T \in \mathcal{PW}_\pi^\infty$, T as defined in (3), is a bounded linear operator from $\mathcal{C}$ into $\mathcal{PW}_\pi^2$. In this paper we only consider stable LTI-systems $\mathcal{C} \to \mathcal{PW}_\pi^2$ with $h_T \in \mathcal{PW}_\pi^\infty$.

A typical example of a LTI system is a bandpass filter, i.e., a system with $\hat{h}_T(\omega) = \mathbf{1}_{[\omega_1,\omega_2]}(\omega)$, $\omega \in [-\pi, \pi]$, where $-\pi \leq \omega_1 < \omega_2 \leq \pi$. $\mathbf{1}_A$ denotes the indicator function of the set A.

For $f \in \mathcal{PW}_\pi^2$ we have, in addition to the convolution integral representation (3) and the frequency domain representation (4), the following convolution sum sampling representation

$$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(k) h_T(t-k), \quad t \in \mathbb{R}, \tag{5}$$

where the series in (5) converges uniformly on the whole real axis. However, the representation (5) is often also used for larger signal spaces, and it is assumed that it is still valid, at least when convergence is treated in the sense of distributions. This, however, is not always the case, as it has been shown in [7], [8]. In [8] convolution sum sampling system representations were analyzed for signals in $\mathcal{PW}_\pi^1$ and non-equidistant sampling patterns, and it was shown that, for every sampling pattern that is a complete interpolating sequence and all $t \in \mathbb{R}$, there exists a stable LTI system T and a signal $f \in \mathcal{PW}_\pi^1$, such that the corresponding convolution sum approximation process diverges at t. In [7] the distributional behavior of (3) and (5) was analyzed for signals in $\mathcal{PW}_\pi^1$, and divergence for certain signals and systems was established.

In many signal processing books and publications distributions are used [1]–[3], [11]–[16]. Distribution have proven to be a helpful tool to get new insights into a problem. However, often distributions are used in a heuristic fashion rather than in a clean mathematical way. Then computations and manipulations lack a proper mathematical justification. In many cases, the performed calculations are nevertheless correct and can be mathematically justified in a distributional setting. However, sometimes, those manipulations are misleading or even wrong, because a proper mathematical justification cannot be established. This situation is very problematic, because in a theory that contains inconsistencies, any statement can be proved true.

In the present work we analyze the convolution sum system representation (5) for signals $f \in \mathcal{C}$ and stable LTI systems $T\colon \mathcal{C} \to \mathcal{PW}_\pi^2$, and show that even for this benign signal space there are convergence problems.

For $f \in \mathcal{C}$, the expressions in the equation

$$\begin{aligned} f_{\text{Ш}}(t) &= f(t) \cdot \text{Ш}(t) \\ &= f(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t-k) \\ &= \sum_{k=-\infty}^{\infty} f(k)\delta(t-k) \end{aligned}$$

are all well-defined distributions in $\mathcal{S}'$. The convergence of the series in the above equation has to be treated in the sense of distributions as described at the end of Section III. Moreover, every function $h_T \in \mathcal{PW}_\pi^\infty$ is a regular distribution in $\mathcal{S}'$. Formally, often the following manipulation

$$\text{“}(Tf)(t) = (f_{\text{Ш}} * h_T)(t) = \sum_{k=-\infty}^{\infty} f(k)h_T(t-k)\text{”} \tag{6}$$

is performed to obtain the system output Tf, and it is assumed that these expressions are meaningful, at least in a distributional setting. However, it is not clear if the convolution $f_{\text{Ш}} * h_T$ is well-defined. A natural approach is to approximate $f_{\text{Ш}}$ by a finite sum

$$f_{\text{Ш},N}(t) = \sum_{k=-N}^{N} f(k)\delta(t-k).$$

$f_{\text{Ш},N}$ is a distribution with finite support. Consequently, the convolution

$$(f_{\text{Ш},N} * h_T)(t) = \sum_{k=-N}^{N} f(k)h_T(t-k)$$

exists, and even defines a classical continuous function. However, a priori it is unclear whether the sequence of functions $\{f_{\text{Ш},N} * h_T\}_{N\in\mathbb{N}}$ converges in $\mathcal{S}'$. In the signal processing literature, starting from classical books like [14], [15], [17], the question of convergence of this sequence is not properly treated.

Example 1. The simplest example of a systems approximation process, having the form (5), is the bandlimited interpolation

$$f_{\text{BL}}(t) = \sum_{k=-\infty}^{\infty} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}, \tag{7}$$

which is obtained when T is the identity (cf. [15, p. 52] and [2, p. 144]). This special case has already been studied in [18], where the distributional behavior was analyzed for signals in $C_0(\mathbb{R})$. We will see in Section VII that the series in (7) is not always well-defined, because there exist signals in $\mathcal{C}$ such that (7) diverges even in $\mathcal{S}'$.

In this work we will analyze the following questions for the practically relevant signal space $\mathcal{C}$:

1) When does the series in (5) converge classically?
2) When does the series in (5) converge in a distributional sense in $\mathcal{S}'$?

Often, the theory of distributions is used, without any restriction on the signal spaces, as a formal technique to justify the convergence and hence the existence of certain mathematical objects, for example, convolution integrals, sums, etc. If it is possible to show divergence phenomena for a rather small signal space that consists only of classically well-defined signals, then it is clear that for any larger signal space that contains these nice signals, we have divergence as well. In our case the space $\mathcal{C}$ consists of continuous signals, vanishing at infinity, that are absolutely, and hence also square integrable. We show that there exist signals in $\mathcal{C}$ for which the series in (5) diverges, both classically, as well as distributionally. Further, we prove for the signal space $\mathcal{C}$ that (5) converges in $\mathcal{S}'$ if and only if it converges classically. Hence, a distributional treatment does not extend the validity of the expression (5), and is not necessary for the signal space $\mathcal{C}$.

Finally, we list several facts about $\mathcal{C}$, which illustrate the nice properties of this space:

1) Since we consider sampling expressions, it is necessary that the sampling operation, i.e., the point evaluation operator, is well-defined. Clearly, this is the case for continuous signals.
2) In distribution theory, generally no assumption is made about the asymptotic behavior of the signals. Nevertheless, we have $\lim_{|t|\to\infty} f(t) = 0$, which is a natural and relevant assumption for applications.
3) The property $f \in L^1(\mathbb{R})$ for signals in $\mathcal{C}$ implies that the Fourier transform integral converges absolutely. Therefore, the Fourier transform is defined straightforwardly by the Fourier transform integral, and the use of the L^2-definition is not necessary. Moreover, $\hat{f}$ is continuous and $\lim_{|\omega|\to\infty} \hat{f}(\omega) = 0$, according to the Riemann–Lebesgue lemma.
4) Since $\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t \le \|f\|_{C_0(\mathbb{R})} \int_{-\infty}^{\infty} |f(t)| \, \mathrm{d}t$, we see that signals in $\mathcal{C}$ are also in $L^2(\mathbb{R})$, i.e., every signal $f \in \mathcal{C}$ has finite energy.

V. Convergence in the Classical Sense

We start with proving the following theorem about pointwise convergence of the system approximation series.

Theorem 1. *Let $h_T \in \mathcal{PW}_\pi^\infty$. Then the series*

$$\sum_{k=-\infty}^{\infty} f(k)h_T(t-k) \tag{8}$$

converges for all $t \in \mathbb{R}$ and all $f \in \mathcal{C}$ if and only if $h_T \in \mathcal{B}_\pi^1$.

Theorem 1 gives a full characterization when we have pointwise convergence for the whole space $\mathcal{C}$ and all $t \in \mathbb{R}$. Interestingly, pointwise convergence for all $f \in \mathcal{C}$ is equivalent to uniform convergence for all $f \in \mathcal{C}$.

Corollary 1. *Let $h_T \in \mathcal{PW}_\pi^\infty$. If the series*

$$\sum_{k=-\infty}^{\infty} f(k)h_T(t-k) \tag{9}$$

converges pointwise for all $t \in \mathbb{R}$ and all $f \in \mathcal{C}$ then it converges globally uniformly for all $f \in \mathcal{C}$.

Proof of Corollary 1. Let $h_T \in \mathcal{PW}_\pi^\infty$, and assume that the series (9) converges for all $t \in \mathbb{R}$ and all $f \in \mathcal{C}$. Then, according to Theorem 1, we have $h_T \in \mathcal{B}_\pi^1$. For $N_2 > N_1$ it follows that

$$\begin{aligned}&\left|\sum_{k=-N_2}^{N_2} f(k)h_T(t-k) - \sum_{k=-N_1}^{N_1} f(k)h_T(t-k)\right| \\ &= \left|\sum_{N_1+1\le|k|\le N_2} f(k)h_T(t-k)\right| \\ &\le \left(\max_{|k|\ge N_1+1}|f(k)|\right)\sum_{k=-\infty}^{\infty}|h_T(t-k)| \\ &\le (1+\pi)\|h_T\|_{\mathcal{B}_\pi^1}\max_{|k|\ge N_1+1}|f(k)|,\end{aligned}$$

because

$$\sum_{k=-\infty}^{\infty}|h_T(t-k)| \le (1+\pi)\int_{-\infty}^{\infty}|h_T(\tau)|\,\mathrm{d}\tau$$

according to Nikol'skiĭ's inequality (see Theorem 6 in the appendix). Hence, for all $\epsilon > 0$ there exists an $N_0 = N_0(\epsilon)$ such that

$$\max_{t\in\mathbb{R}}\left|\sum_{k=-N_2}^{N_2} f(k)h_T(t-k) - \sum_{k=-N_1}^{N_1} f(k)h_T(t-k)\right| < \epsilon$$

for all $N_1, N_2 \ge N_0(\epsilon)$. □

For the proof of Theorem 1 we need two lemmas.

Lemma 1. *Let $a \in c_0$. Then there exists an $f_a \in \mathcal{C}$ such that*

$$f_a(k) = a_k, \quad k \in \mathbb{Z},$$

and

$$\|f_a\|_{\mathcal{C}} \le \|a\|_{c_0}.$$

Proof. For $k \in \mathbb{Z}$, let $l_k = 2^{-|k|}/3$ and define

$$d_k(t) = \begin{cases} 0, & |t-k| \ge l_k, \\ 1 - l_k^{-1}|t-k|, & |t-k| < l_k. \end{cases}$$

Let

$$f_a(t) = \sum_{k=-\infty}^{\infty} a_k d_k(t), \quad t \in \mathbb{R}. \tag{10}$$

Since all d_k, $k \in \mathbb{Z}$, have pairwise disjoint support, the convergence of the series in (10) is obvious. Clearly, we have $f_a(k) = a_k$ for all $k \in \mathbb{Z}$. Further, we have

$$\begin{aligned}\int_{-\infty}^{\infty}|f_a(t)|\,\mathrm{d}t &\le \|a\|_{c_0}\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|d_k(t)|\,\mathrm{d}t \\ &= \|a\|_{c_0},\end{aligned}$$

because

$$\begin{aligned}\int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty}|d_k(t)|\,\mathrm{d}t &= \sum_{k=-\infty}^{\infty}\int_{-\infty}^{\infty} d_k(t)\,\mathrm{d}t \\ &= \sum_{k=-\infty}^{\infty} l_k \\ &= \frac{1}{3}\sum_{k=-\infty}^{\infty}\frac{1}{2^{|k|}} \\ &= 1.\end{aligned}$$

Moreover, we have

$$|f_a(t)| \le \sum_{k=-\infty}^{\infty}|a_k||d_k(t)| \le \|a\|_{c_0}$$

and

$$\lim_{|t|\to\infty}|f_a(t)| = 0.$$

Hence, it follows that $f_a \in \mathcal{C}$ with $\|f_a\|_{\mathcal{C}} \le \|a\|_{c_0}$. □

The second needed lemma is stated next.

Lemma 2. *Let $h_T \in \mathcal{PW}_\pi^\infty$, $t \in \mathbb{R}$, and*

$$T_{N,t}f = (T_N f)(t) = \sum_{k=-N}^{N} f(k)h_T(t-k).$$

Then we have

$$\|T_{N,t}\| = \sup_{\|f\|_{\mathcal{C}}\le 1}|(T_N f)(t)| = \sum_{k=-N}^{N}|h_T(t-k)|.$$

Proof. Let $h_T \in \mathcal{PW}_\pi^\infty$ and $t \in \mathbb{R}$ be arbitrary but fixed. We have

$$\begin{aligned}|(T_N f)(t)| &\le \sum_{k=-N}^{N}|f(k)||h_T(t-k)| \\ &\le \|f\|_{\mathcal{C}}\sum_{k=-N}^{N}|h_T(t-k)|,\end{aligned}$$

which shows that

$$\|T_{N,t}\| \le \sum_{k=-N}^{N}|h_T(t-k)|. \tag{11}$$

For $N \in \mathbb{N}$, let

$$a_k^{(N)} = \begin{cases} \operatorname{sgn}(h_T(t-k)), & |k| \le N, \\ 0, & |k| > N. \end{cases}$$

According to Lemma 1, there exists a function $g_N \in \mathcal{C}$ such that $g_N(k) = a_k^{(N)}$, $k \in \mathbb{Z}$, and $\|g_N\|_{\mathcal{C}} \le 1$. We have

$$\begin{aligned}(T_N g_N)(t) &= \sum_{k=-N}^{N} g_N(k)h_T(t-k) \\ &= \sum_{k=-N}^{N}\operatorname{sgn}(h_T(t-k))h_T(t-k) \\ &= \sum_{k=-N}^{N}|h_T(t-k)|,\end{aligned}$$

which shows that

$$\|T_{N,t}\| \geq \sum_{k=-N}^{N} |h_T(t-k)|. \tag{12}$$

Combining (11) and (12) gives the assertion of the lemma. □

Now we are in the position to prove Theorem 1.

Proof of Theorem 1. "⇒": Let $t \in \mathbb{R}$ be arbitrary but fixed. Since the series (8) converges for all $f \in \mathcal{C}$, it follows that

$$\sup_{N\in\mathbb{N}} |T_{N,t} f| < \infty$$

for all $f \in \mathcal{C}$. Hence, according to the Banach–Steinhaus theorem (see Theorem 5 in the appendix), we have

$$\sup_{N\in\mathbb{N}} \|T_{N,t}\| < \infty,$$

which in turn implies that

$$\sum_{k=-\infty}^{\infty} |h_T(t-k)| < \infty,$$

according to Lemma 2.

From

$$\sum_{k=-\infty}^{\infty} |h_T(t-k)| < \infty$$

for all $t \in \mathbb{R}$ we can conclude that

$$\sum_{k=-\infty}^{\infty} |h_T(k)| = C_1 < \infty$$

and

$$\sum_{k=-\infty}^{\infty} \left| h_T\left(\frac{1}{2}-k\right)\right| = C_2 < \infty,$$

and consequently that

$$\sum_{k=-\infty}^{\infty} \left| h_T\left(\frac{k}{2}\right)\right| = C_1 + C_2 < \infty.$$

Let

$$\hat{g}(\omega) = \begin{cases} \omega/\pi + 2, & -2\pi \leq \omega < -\pi, \\ 1, & -\pi \leq \omega \leq \pi, \\ -\omega/\pi + 2, & \pi \leq \omega < 2\pi. \end{cases}$$

Then we have

$$h_T(t) = \sum_{k=-\infty}^{\infty} h_T\left(\frac{k}{2}\right) g\left(t - \frac{k}{2}\right), \quad t \in \mathbb{R}. \tag{13}$$

Further, due to the definition of $\hat{g}$, we have

$$|g(t)| \leq \frac{C_3}{1+|t|^2},$$

and it follows that

$$\sum_{l=-\infty}^{\infty} |g(t-l)| \leq C_3 \sum_{l=-\infty}^{\infty} \frac{1}{1+|t-l|^2} \leq C_4,$$

where the constant C_4 is independent of t. From (13) we see that

$$|h_T(t-l)| \leq \sum_{k=-\infty}^{\infty} \left| h_T\left(\frac{k}{2}\right)\right| \left| g\left(t-l-\frac{k}{2}\right)\right|$$

and it follows

$$\sum_{l=-\infty}^{\infty} |h_T(t-l)| \leq \sum_{k=-\infty}^{\infty} \left| h_T\left(\frac{k}{2}\right)\right| \sum_{l=-\infty}^{\infty} \left| g\left(t-l-\frac{k}{2}\right)\right|.$$

Since

$$\sum_{l=-\infty}^{\infty} \left| g\left(t-l-\frac{k}{2}\right)\right| \leq C_4,$$

we obtain

$$\sum_{l=-\infty}^{\infty} |h_T(t-l)| \leq C_4 \sum_{k=-\infty}^{\infty} \left| h_T\left(\frac{k}{2}\right)\right| = C_4(C_1+C_2). \tag{14}$$

The right-hand side of (14) is independent of t and thus it follows that

$$\int_{-\infty}^{\infty} |h_T(t)| \, \mathrm{d}t = \int_0^1 \sum_{l=-\infty}^{\infty} |h_T(t-l)| \, \mathrm{d}t \leq C_4(C_1+C_2),$$

which shows that $h_T \in \mathcal{B}_\pi^1$.

"⇐": If $h_T \in \mathcal{B}_\pi^1$, then we have

$$\sum_{k=-\infty}^{\infty} |h_T(t-k)| \leq (1+\pi) \int_{-\infty}^{\infty} |h_T(\tau)| \, \mathrm{d}\tau, \tag{15}$$

according to Nikol'skiĭ's inequality (see Theorem 6 in the appendix), and it follows that

$$\begin{aligned}\sum_{k=-\infty}^{\infty} |f(k) h_T(t-k)| &\leq \|f\|_{\mathcal{C}} \sum_{k=-\infty}^{\infty} |h_T(t-k)| \\ &\leq \|f\|_{\mathcal{C}} (1+\pi) \int_{-\infty}^{\infty} |h_T(\tau)| \, \mathrm{d}\tau,\end{aligned}$$

which shows that the series (8) is absolutely convergent and hence convergent. □

VI. Size of the Divergence Set I

From Theorem 1 we obtain a corollary about the size of the set of stable LTI systems for which divergence can occur. To state the theorem, we need to introduce the concept of Baire categories.

A subset $\mathcal{M}$ of a Banach space $\mathcal{X}$ is said to be nowhere dense in $\mathcal{X}$ if the interior of the closure of $\mathcal{M}$ is empty. $\mathcal{M}$ is said to be of first category (or meager) if $\mathcal{M}$ is the countable union of sets each of which is nowhere dense in $\mathcal{X}$. $\mathcal{M}$ is said to be of second category (or nonmeager) if is not of the first category. The complement of a set of the first category is called a residual set. Topologically, sets of first category may be considered as "small". Accordingly, residual sets, being the complements of sets of first category, can be considered as "large". In a complete metric space any residual set is dense and a set of second category, due to Baire's theorem [19].

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The final version of record is available at http://dx.doi.org/10.1109/TSP.2018.2865435

Corollary 2. *The set of all stable LTI systems $h_T \in \mathcal{PW}_\pi^\infty$, for which there exists an $f \in \mathcal{C}$ and a $t \in \mathbb{R}$ such that*

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| = \infty, \tag{16}$$

is a residual set in $\mathcal{PW}_\pi^\infty$.

Moreover, for each h_T from this residual set, the set of signals $f \in \mathcal{C}$, for which there exists a $t \in \mathbb{R}$ such that (16) *is true, is a residual set in $\mathcal{C}$.*

Corollary 2 shows that divergence of the series (16) is not a rare event that occurs only for few signals and systems but, from a topological point of view, a rather frequent event.

Proof of Corollary 2. First, we establish the fact that the set

$$\mathcal{D} = \left\{ h_T \in \mathcal{PW}_\pi^\infty \colon h_T \notin \mathcal{B}_\pi^1 \right\}$$

is a residual set in $\mathcal{PW}_\pi^\infty$. Since $\mathcal{B}_\pi^1 \subset \mathcal{PW}_\pi^\infty$, it is sufficient to show that $\mathcal{B}_\pi^1$ is a set of first category in $\mathcal{PW}_\pi^\infty$. We assume that $\mathcal{B}_\pi^1$ is not a set of first category, i.e., that $\mathcal{B}_\pi^1$ is a set of second category, and construct a contradiction. For $h_T \in \mathcal{PW}_\pi^\infty$, consider the functionals

$$F_N(h_T) = \int_{-N}^{N} |h_T(t)| \, \mathrm{d}t.$$

For each $N \in \mathbb{N}$, $F_N \colon \mathcal{PW}_\pi^\infty \to \mathbb{R}$ is a continuous functional. We have

$$\lim_{N\to\infty} F_N(\underline{h}_T) < \infty$$

for $\underline{h}_T \in \mathcal{B}_\pi^1$. Hence, the generalized uniform boundedness principle (see Theorem 4 in the appendix for more details) implies that there exists an open ball $\mathcal{U}_{\tilde{\epsilon}}(\tilde{h}_T)$ in $\mathcal{PW}_\pi^\infty$ and a constant C_5 such that

$$\lim_{N\to\infty} F_N(h_T) \le C_5$$

for all $h_T \in \mathcal{U}_{\tilde{\epsilon}}(\tilde{h}_T)$. That is, we have

$$\int_{-\infty}^{\infty} h_T(t) \, \mathrm{d}t \le C_5$$

for all $h_T \in \mathcal{U}_{\tilde{\epsilon}}(\tilde{h}_T)$. For all $h_T^{(1)} \in \mathcal{PW}_\pi^\infty$ with $\|h_T^{(1)}\|_{\mathcal{PW}_\pi^\infty} < \tilde{\epsilon}$ we have $h_T^{(2)} := \tilde{h}_T + h_T^{(1)} \in \mathcal{U}_{\tilde{\epsilon}}(\tilde{h}_T)$. It follows that

$$\int_{-\infty}^{\infty} |h_T^{(2)}(t)| \, \mathrm{d}t \le C_5,$$

and consequently that

$$\begin{aligned} \int_{-\infty}^{\infty} |h_T^{(1)}(t)| \, \mathrm{d}t &= \int_{-\infty}^{\infty} |h_T^{(2)}(t) - \tilde{h}_T(t)| \, \mathrm{d}t \\ &\le \int_{-\infty}^{\infty} |h_T^{(2)}(t)| \, \mathrm{d}t + \int_{-\infty}^{\infty} |\tilde{h}_T(t)| \, \mathrm{d}t \\ &\le 2C_5 \end{aligned} \tag{17}$$

for all $h_T^{(1)} \in \mathcal{PW}_\pi^\infty$ with $\|h_T^{(1)}\|_{\mathcal{PW}_\pi^\infty} < \tilde{\epsilon}$. Now let $h_T^{(3)} \in \mathcal{PW}_\pi^\infty$ be arbitrary and set

$$h_T^{(4)} = \frac{\tilde{\epsilon}}{2\|h_T^{(3)}\|_{\mathcal{PW}_\pi^\infty}} h_T^{(3)}.$$

Then we have $\|h_T^{(4)}\|_{\mathcal{PW}_\pi^\infty} = \tilde{\epsilon}/2 < \tilde{\epsilon}$. It follows that

$$\begin{aligned} \int_{-\infty}^{\infty} |h_T^{(3)}(t)| \, \mathrm{d}t &= \frac{2\|h_T^{(3)}\|_{\mathcal{PW}_\pi^\infty}}{\tilde{\epsilon}} \int_{-\infty}^{\infty} |h_T^{(4)}(t)| \, \mathrm{d}t \\ &\le \frac{4C_5 \|h_T^{(3)}\|_{\mathcal{PW}_\pi^\infty}}{\tilde{\epsilon}}, \end{aligned} \tag{18}$$

for all $h_T^{(3)} \in \mathcal{PW}_\pi^\infty$, where we used (17) in the last inequality. However, for $h_{\text{LP}} = \text{sinc} \in \mathcal{PW}_\pi^\infty$, we have

$$\int_{-\infty}^{\infty} |h_{\text{LP}}(t)| \, \mathrm{d}t = \infty,$$

which is a contradiction to (18). This shows that $\mathcal{B}_\pi^1$ is a set of first category in $\mathcal{PW}_\pi^\infty$, or, equivalently, that $\mathcal{D}$ is a residual set in $\mathcal{PW}_\pi^\infty$.

Next, we show that $h_T \notin \mathcal{B}_\pi^1$ implies that there exists an $f \in \mathcal{C}$ and $t \in \mathbb{R}$ such that (16) holds. Let $h_T \notin \mathcal{B}_\pi^1$ be arbitrary but fixed. Then there exists a $t_* \in \mathbb{R}$ such that

$$\sum_{k=-\infty}^{\infty} |h_T(t_* - k)| = \infty,$$

and Lemma 2 shows that $\sup_{N\in\mathbb{N}} \|T_{N,t_*}\| = \infty$. Using the Banach–Steinhaus theorem (see Theorem 5 in the appendix) we obtain that there exists a signal $f_* \in \mathcal{C}$ such that $\sup_{N\in\mathbb{N}} |T_{N,t_*} f_*| = \infty$.

Now we prove the second statement. For every $h_{T_*} \in \mathcal{D}$, there exists a signal $f_* \in \mathcal{C}$ and a $t_* \in \mathbb{R}$ such that

$$\limsup_{N\to\infty} \left| \sum_{k=-N}^{N} f_*(k) h_{T_*}(t_* - k) \right| = \infty. \tag{19}$$

For the functional $T_{*N,t_*} \colon \mathcal{C} \to \mathbb{C}$, defined by

$$T_{*N,t_*} f = \sum_{k=-N}^{N} f(k) h_{T_*}(t_* - k)$$

we therefore have $\sup_{N\in\mathbb{N}} \|T_{*N,t_*}\| = \infty$. Application of the Banach–Steinhaus theorem (see Theorem 5 in the appendix) shows that the set of signals $f \in \mathcal{C}$ for which we have $\sup_{N\in\mathbb{N}} |T_{*N,t_*} f| = \infty$ is a residual set. $\square$

VII. Convergence in the Distributional Sense

In Section V we have seen that the convolution sum (5) diverges pointwise for certain signals $f \in \mathcal{C}$. In this section we study the distributional behavior of

$$\sum_{k=-\infty}^{\infty} f(k) h_T(t-k) \tag{20}$$

in $\mathcal{S}'$ for $h_T \in \mathcal{PW}_\pi^\infty$ and $f \in \mathcal{C}$. Let

$$(T_N f)(t) = \sum_{k=-N}^{N} f(k) h_T(t-k).$$

According to the definition of distributional convergence, the sequence $\{T_N f\}_{N\in\mathbb{N}}$ converges in $\mathcal{S}'$ if and only if

$$\left\{ \int_{-\infty}^{\infty} (T_N f)(t) \phi(t) \, \mathrm{d}t \right\}_{N\in\mathbb{N}} \tag{21}$$

converges for all $\phi \in \mathcal{S}$. We consider the continuous linear functional $G_{N,\phi,h_T}\colon \mathcal{C} \to \mathbb{C}$, defined by

$$\begin{aligned} G_{N,\phi,h_T} f &= \int_{-\infty}^{\infty} (T_N f)(t)\phi(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \sum_{k=-N}^{N} f(k)h_T(t-k)\phi(t)\,\mathrm{d}t \\ &= \sum_{k=-N}^{N} f(k) \underbrace{\int_{-\infty}^{\infty} h_T(t-k)\phi(t)\,\mathrm{d}t}_{=c_k(h_T,\phi)}. \end{aligned}$$

Hence $\{T_N f\}_{N\in\mathbb{N}}$ converges in $\mathcal{S}'$ if and only if $\{G_{N,\phi,h_T} f\}_{N\in\mathbb{N}}$ converges for all $\phi \in \mathcal{S}$. Using the same calculations as in the proof of Lemma 2, it can be shown that

$$\|G_{N,\phi,h_T}\| = \sum_{k=-N}^{N} |c_k(h_T,\phi)|. \tag{22}$$

This observation leads to the next lemma.

Lemma 3. *Let $h_T \in \mathcal{PW}_\pi^\infty$. Then*

$$\sum_{k=-\infty}^{\infty} f(k)h_T(t-k) \tag{23}$$

converges in $\mathcal{S}'$ for all $f \in \mathcal{C}$ if and only if for all $\phi \in \mathcal{S}$ we have

$$\sum_{k=-\infty}^{\infty} |c_k(h_T,\phi)| < \infty. \tag{24}$$

Proof. "$\Rightarrow$": Let $h_T \in \mathcal{PW}_\pi^\infty$ and assume that (23) converges in $\mathcal{S}'$ for all $f \in \mathcal{C}$. Then, for all $f \in \mathcal{C}$, the sequence $\{G_{N,\phi,h_T} f\}_{N\in\mathbb{N}}$ converges for all $\phi \in \mathcal{S}$. This implies that

$$\sup_{N\in\mathbb{N}} |G_{N,\phi,h_T} f| < \infty$$

for all $\phi \in \mathcal{S}$ and all $f \in \mathcal{C}$. Using the Banach–Steinhaus theorem (see Theorem 5 in the appendix), we see that

$$\sup_{N\in\mathbb{N}} \|G_{N,\phi,h_T}\| < \infty$$

for all $\phi \in \mathcal{S}$, which in turn implies, using (22), that

$$\sum_{k=-\infty}^{\infty} |c_k(h_T,\phi)| < \infty.$$

for all $\phi \in \mathcal{S}$.

"$\Leftarrow$": Let $h_T \in \mathcal{PW}_\pi^\infty$. Further, let $\phi \in \mathcal{S}$ and $f \in \mathcal{C}$ be arbitrary but fixed. For $N > M$ we have

$$\begin{aligned} |G_{N,\phi,h_T} f - G_{M,\phi,h_T} f| &= \left| \sum_{M<|k|\le N} f(k)c_k(h_T,\phi) \right| \\ &\le \|f\|_{\mathcal{C}} \sum_{M<|k|} |c_k(h_T,\phi)|. \end{aligned}$$

Since, by assumption, we have (24), and consequently that

$$\lim_{M\to\infty} \sum_{M<|k|} |c_k(h_T,\phi)| = 0,$$

we see that $\{G_{N,\phi,h_T} f\}_{N\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{R}$, and therefore convergent. Because $\phi \in \mathcal{S}$ and $f \in \mathcal{C}$ were chosen arbitrarily, and because $\mathcal{S}'$ is closed under convergence [10, p. 104], we have established the convergence of (23) in $\mathcal{S}'$ for all $f \in \mathcal{C}$. $\square$

Using the previous lemma, we can prove Theorem 2, which gives a complete characterization of the convergence of the series (20) in $\mathcal{S}'$ for all $f \in \mathcal{C}$. Interestingly, the condition $h_T \in \mathcal{B}_\pi^1$ is exactly the same as in the case of pointwise divergence.

Theorem 2. *Let $h_T \in \mathcal{PW}_\pi^\infty$. Then the series*

$$\sum_{k=-\infty}^{\infty} f(k)h_T(t-k) \tag{25}$$

converges in $\mathcal{S}'$ for all $f \in \mathcal{C}$ if and only if $h_T \in \mathcal{B}_\pi^1$.

Proof. From Lemma 3 we know that (25) converges in $\mathcal{S}'$ for all $f \in \mathcal{C}$ if and only if we have (24) for all $\phi \in \mathcal{S}$. Hence, it suffices to prove that we have (24) for all $\phi \in \mathcal{S}$ if and only if $h_T \in \mathcal{B}_\pi^1$.

"$\Rightarrow$": Assume that (24), i.e.

$$\sum_{k=-\infty}^{\infty} |c_k(h_T,\phi)| < \infty \tag{26}$$

is true for all $\phi \in \mathcal{S}$. Let $\tau \in \mathbb{R}$ be arbitrary but fixed, and choose a $\phi_* \in \mathcal{S}$ such that $\hat{\phi}_*(\omega) = \mathrm{e}^{-i\omega\tau}$, $|\omega| \le \pi$. Such a function can be easily constructed in the frequency domain by multiplying an infinitely often differentiable function $\hat{u}(\omega)$ that equals one for $|\omega| \le \pi$ with $\mathrm{e}^{-i\omega\tau}$. Then we have

$$\begin{aligned} c_k(h_T,\phi_*) &= \int_{-\infty}^{\infty} h_T(t-k)\phi_*(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{h}_T(\omega)\,\mathrm{e}^{i\omega(t-k)}\,\mathrm{d}\omega\,\phi_*(t)\,\mathrm{d}t \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{h}_T(\omega)\,\mathrm{e}^{-i\omega k}\int_{-\infty}^{\infty} \phi_*(t)\,\mathrm{e}^{i\omega t}\,\mathrm{d}t\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{h}_T(\omega)\,\mathrm{e}^{-i\omega k}\,\hat{\phi}_*(-\omega)\,\mathrm{d}\omega \\ &= \frac{1}{2\pi}\int_{-\pi}^{\pi} \hat{h}_T(\omega)\,\mathrm{e}^{-i\omega k}\,\mathrm{e}^{i\omega\tau}\,\mathrm{d}\omega \\ &= h_T(\tau-k) \end{aligned} \tag{27}$$

for all $k \in \mathbb{Z}$, and it follows from (26) that

$$\sum_{k=-\infty}^{\infty} |h_T(\tau-k)| < \infty.$$

Since $\tau \in \mathbb{R}$ was arbitrary, it follows by the same line of reasoning as in the proof of Theorem 1 that $h_T \in \mathcal{B}_\pi^1$.

"$\Leftarrow$": Now assume that $h_T \in \mathcal{B}^1_\pi$. Then we have

$$\begin{aligned}\sum_{k=-\infty}^{\infty} |c_k(h_T,\phi)| &\leq \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} |h_T(t-k)||\phi(t)| \, \mathrm{d}t \\ &= \int_{-\infty}^{\infty} |\phi(t)| \sum_{k=-\infty}^{\infty} |h_T(t-k)| \, \mathrm{d}t \\ &\leq \int_{-\infty}^{\infty} |\phi(t)|(1+\pi)\|h_T\|_{\mathcal{B}^1_\pi} \, \mathrm{d}t \\ &= (1+\pi)\|h_T\|_{\mathcal{B}^1_\pi}\|\phi\|_1 < \infty,\end{aligned}$$

where we again used Nikol'skiĭ's inequality (see Theorem 6 in the appendix) in the second to last line. □

We present several examples of stable LTI systems T, for which we have $h_T \notin \mathcal{B}^1_\pi$, next. For those systems there exists a signal $f \in \mathcal{C}$ such that the series (25) diverges even in a distributional setting.

Example 2. Let $0 < \sigma \leq \pi$. For the ideal low-pass filter with $\hat{h}_{\mathrm{LP}}(\omega) = \mathbf{1}_{[-\sigma,\sigma]}(\omega)$, $\omega \in [-\pi,\pi]$ we have $h_{\mathrm{LP}} = \frac{\sin(\sigma\,\cdot)}{\pi\,\cdot} \notin \mathcal{B}^1_\pi$.

Example 3. For the Hilbert transform with $\hat{h}_{\mathrm{H}}(\omega) = -i\,\mathrm{sgn}(\omega)$ we have $h_{\mathrm{H}} = \frac{1-\cos(\pi\,\cdot)}{\pi\,\cdot} \notin \mathcal{B}^1_\pi$.

Choosing $\sigma = \pi$ in Example 2 gives, as a special case, the Shannon sampling series

$$\sum_{k=-\infty}^{\infty} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{28}$$

For the Shannon sampling series it is easy to explicitly construct a signal $f \in \mathcal{C}$ such that (28) diverges for all $t \in \mathbb{R}\setminus\mathbb{Z}$. We follow a construction which was presented in [20]. For $k \in \mathbb{Z}$, let

$$a_k = \begin{cases} \frac{(-1)^k}{\log 1+k}, & k \geq 1, \\ 0, & k \leq 0. \end{cases}$$

In [20] it was proved that

$$\lim_{N\to\infty} \left| \sum_{k=-N}^{N} a_k \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty \tag{29}$$

for all $t \in \mathbb{R}\setminus\mathbb{Z}$. Using the procedure that was used in the proof of Lemma 1, we can construct a signal $f_* \in \mathcal{C}$ with $f(k) = a_k$, $k \in \mathbb{Z}$. Clearly, for this f_* we have the divergence

$$\lim_{N\to\infty} \left| \sum_{k=-N}^{N} f_*(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty$$

for all $t \in \mathbb{R}\setminus\mathbb{Z}$, because of (29).

The previous results and examples show that there are stable LTI systems T and signals $f \in \mathcal{C}$ such that the sampling representation (20) of the system diverges even in $\mathcal{S}'$. This shows that using a distributional approach here does not circumvent the convergence problems. Moreover, the next theorem shows that the sampling representation (20) converges in the classical pointwise sense for all $f \in \mathcal{C}$ if and only if it converges in the distributional sense for all $f \in \mathcal{C}$. Hence, with respect to convergence, nothing is gained by using a distributional setting.

Theorem 3. *Let $h_T \in \mathcal{PW}^\infty_\pi$. The series*

$$\sum_{k=-\infty}^{\infty} f(k) h_T(t-k) \tag{30}$$

converges in $\mathcal{S}'$ for all $f \in \mathcal{C}$ if and only if it converges pointwise for all $t \in \mathbb{R}$ and all $f \in \mathcal{C}$.

Proof. "$\Rightarrow$": According to Theorem 2 we have $h_T \in \mathcal{B}^1_\pi$. Hence, by Theorem 1 the series (30) converges classically for all $t \in \mathbb{R}$ and all $f \in \mathcal{C}$. "$\Leftarrow$": According to Theorem 1 we have $h_T \in \mathcal{B}^1_\pi$, and Theorem 2 gives the convergence of the series (30) in $\mathcal{S}'$ for all $f \in \mathcal{C}$.. □

The previous result can even be strengthened. Distributional convergence for all $f \in \mathcal{C}$ is equivalent to global uniform convergence for all $f \in \mathcal{C}$.

Corollary 3. *Let $h_T \in \mathcal{PW}^\infty_\pi$. If the series*

$$\sum_{k=-\infty}^{\infty} f(k) h_T(t-k)$$

converges in $\mathcal{S}'$ for all $f \in \mathcal{C}$ then it converges globally uniformly for all $f \in \mathcal{C}$.

Proof. This is a direct consequence of Theorem 3 and Corollary 1. □

VIII. Size of the Divergence Set II

Similar to Section VI, where we analyzed the size of the set of stable LTI systems and signals for which we have divergence in the classical pointwise sense, we can use the findings from the previous section to derive a result for the distributional case.

Corollary 4. *The set of all stable LTI systems $h_T \in \mathcal{PW}^\infty_\pi$, for which there exists an $f \in \mathcal{C}$ and a $\phi \in \mathcal{S}$ such that*

$$\limsup_{N\to\infty} |G_{N,\phi,h_T} f| = \infty, \tag{31}$$

is a residual set in $\mathcal{PW}^\infty_\pi$.

Moreover, for each h_T from this residual set, the set of signals $f \in \mathcal{C}$ for which (31) *is true for some $\phi \in \mathcal{S}$, is a residual set in $\mathcal{C}$.*

Proof. In the proof of Corollary 2, we have shown that the set

$$\mathcal{D} = \left\{ h_T \in \mathcal{PW}^\infty_\pi : h_T \notin \mathcal{B}^1_\pi \right\}$$

is a residual set in $\mathcal{PW}^\infty_\pi$. By showing that $h_T \notin \mathcal{B}^1_\pi$ implies that (31) holds for some $f \in \mathcal{C}$ and some $\phi \in \mathcal{S}$, we complete the proof of the first statement. Let $h_T \notin \mathcal{B}^1_\pi$ be arbitrary but fixed. Hence, there exists a $t_* \in \mathbb{R}$ such that

$$\sum_{k=-\infty}^{\infty} |h_T(t_*-k)| = \infty.$$

Let $\phi_* \in \mathcal{S}$ be such that $\hat{\phi}_* = \mathrm{e}^{-i\omega t_*}$, $|\omega| \leq \pi$. By the same calculation as in (27), we see that

$$\sum_{k=-\infty}^{\infty} |c_k(h_T,\phi_*)| = \infty.$$

Thus, according to (22), we have

$$\sup_{N\in\mathbb{N}} \|G_{N,\phi_*,h_T}\| = \infty,$$

and the application of the Banach–Steinhaus theorem (see Theorem 5 in the appendix) shows that there exists an $f_* \in \mathcal{C}$ such that

$$\limsup_{N\to\infty} |G_{N,\phi_*,h_T} f_*| = \infty.$$

Now we prove the second statement. Let $h_{T_*} \in \mathcal{PW}_\pi^\infty$ be a function from this residual set. Then there exists an $f_* \in \mathcal{C}$ and an $\phi_* \in \mathcal{S}$ such that

$$\limsup_{N\to\infty} \left|G_{N,\phi_*,h_{T_*}} f_*\right| = \infty$$

for some $\phi_* \in \mathcal{S}$. The Banach–Steinhaus theorem (see Theorem 5 in the appendix) implies that that the set of signals $f \in \mathcal{C}$ for which

$$\limsup_{N\to\infty} \left|G_{N,\phi_*,h_{T_*}} f\right| = \infty$$

is a residual set. □

IX. Discussion

We have shown that the convolution sum (5) diverges pointwise for certain systems and signals. Since, for the considered signal space $\mathcal{C}$, classical pointwise convergence is equivalent to convergence in a distributional sense, this result also implies the divergence of the convolution sum (5) in the sense of distributions for certain systems and signals. Thus, for the signal space $\mathcal{C}$, the convolution sum cannot be defined as the limit of the sequence

$$\left\{ \sum_{k=-N}^{N} f(k) h_T(t-k) \right\}_{N\in\mathbb{N}}$$

in $\mathcal{S}'$. However, our theory makes no statement whether there maybe is a different way to define a "generalized" convolution sum with meaningful properties as an element of $\mathcal{S}'$. Nevertheless, the results in this paper show that the formal use of distributions, as done in many signal processing books, is mathematically not justified for the signal space $\mathcal{C}$.

Interestingly, there is a big difference between the convergence behavior of the time-domain convolution integral system representation (3), the frequency domain representation (4), and the convolution sum representation (5). While the two former are absolutely convergent, the latter can be divergent, even when treated in a distributional setting. Further, it is worth noting that in a topological sense the divergence of the convolution sum (5) holds for a large set of systems and signals, as stated in Corollaries 2 and 4.

Appendix

Basics of Functional Analysis

Since we employ several concepts from functional analysis, we summarize the most important facts here.

A linear operator $T\colon \mathcal{X}_1 \to \mathcal{X}_2$, mapping from a Banach space $\mathcal{X}_1$ in to a Banach space $\mathcal{X}_2$, is called bounded if there exists a constant $C < \infty$ such that

$$\|Tx\|_{\mathcal{X}_2} \le C\|x\|_{\mathcal{X}_1} \tag{32}$$

for all $x \in \mathcal{X}_1$. The smallest possible constant in (32) is called operator norm and denoted by $\|T\|_{\mathcal{X}_1\to\mathcal{X}_2}$. It can be shown that

$$\|T\|_{\mathcal{X}_1\to\mathcal{X}_2} = \sup_{\substack{x\in\mathcal{X}_1\\ \|x\|_{\mathcal{X}_1}\le 1}} \|Tx\|_{\mathcal{X}_2}.$$

Let $\mathcal{X}$ be a Banach space. By

$$\mathcal{U}_{\tilde\epsilon}(\tilde x) = \{x \in \mathcal{X}\colon \|x - \tilde x\|_{\mathcal{X}} < \tilde\epsilon\}$$

we denote the open ball at x with radius ϵ. A key result in functional analysis is the uniform boundedness theorem [21, Theorem 16.2, p. 45].

Theorem 4 (Generalized Uniform Boundedness Theorem)**.** *Let $\mathcal{X}$ be a Banach space and $\underline{\mathcal{X}}$ a set of second category in $\mathcal{X}$. Further, let $\mathcal{F}$ be a set of continuous functions mapping from $\mathcal{X}$ into $\mathbb{R}$, and satisfying*

$$\sup_{F\in\mathcal{F}} F(\underline{x}) < \infty \tag{33}$$

for all $\underline{x} \in \underline{\mathcal{X}}$. Then there exists an open ball $\mathcal{U}_{\tilde\epsilon}(\tilde x)$ in $\mathcal{X}$ and a constant $C < \infty$ such that

$$F(x) \le C$$

for all $x \in \mathcal{U}_{\tilde\epsilon}(\tilde x)$ and all $F \in \mathcal{F}$.

We stated here a slightly more general version of the theorem, where we require (33) to hold only for a set of second category instead of the whole space. Nevertheless, the proof of Theorem 4 is similar to the proof in [21, Theorem 16.2, p. 45]. The Banach–Steinhaus theorem [22, p. 98] can be seen as a consequence of the previous theorem.

Theorem 5 (Banach–Steinhaus Theorem)**.** *Let $\mathcal{X}$ be a Banach space, $\mathcal{Y}$ a normed linear space, and $\{T_N\}_{N\in\mathbb{N}}$ a sequence of bounded linear operators mapping from $\mathcal{X}$ into $\mathcal{Y}$. Then either there exists a $C < \infty$ such that*

$$\sup_{N\in\mathbb{N}} \|T_N\| \le C,$$

or

$$\sup_{N\in\mathbb{N}} \|T_N f\| = \infty$$

for all f belonging to some residual set in $\mathcal{X}$.

Nikol'skiǐ's Inequality

We state a slightly simplified version Nikol'skiǐ's inequality [9, p. 49], which is sufficient for our purposes.

Theorem 6 (Nikol'skiǐ's Inequality)**.** *Let $1 \le p \le \infty$. Then we have*

$$\|f\|_p \le \sup_{t\in\mathbb{R}} \left(\sum_{k=-\infty}^{\infty} |f(t-k)|^p \right)^{\frac{1}{p}} \le (1+\pi)\|f\|_p$$

for all $f \in \mathcal{B}_\pi^p$.

ACKNOWLEDGMENT

Holger Boche wants to thank Bernd Meinerzhagen for discussions on distributions, in particular during the German Research Foundation meeting in October 2017. Parts of this work were motivated by these discussions. Holger Boche would also like to thank Helmut Schreiber for introducing him to the theory of distributions and for his long-time support.

This work was supported by the Gottfried Wilhelm Leibniz Programme of the German Research Foundation (DFG) under grant BO 1734/20-1.

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