Non-Existence of Convolution Sum System Representations

Holger Boche, *Fellow, IEEE,* Ullrich J. Mönich, *Senior Member, IEEE*, and Bernd Meinerzhagen, *Member, IEEE*

***Abstract*—Convolution sum system representations are commonly used in signal processing. It is known that the convolution sum, treated as the limit of its partial sums, can be divergent for certain continuous signals and stable linear time-invariant (LTI) systems, even when the convergence of the partial sums is treated in a distributional setting. In this paper we ask a far more general question: is it at all possible to define a generalized convolution sum with natural properties that works for all absolutely integrable continuous signals that vanish at infinity and all stable LTI systems? We prove that the answer is "no". Further, for certain subspaces, we give a sufficient and necessary condition for uniform convergence. Finally, we discuss the implications of our results on the effectiveness of window functions in the convolution sum.**

***Index Terms*—Linear time-invariant system, continuous signal, convolution sum, distribution, non-existence, window function**

I. Introduction

LINEAR time-invariant (LTI) systems, a key tool in signal processing, are widely used both for theory and applications [1]–[4]. For bandlimited input signals f with finite energy and stable LTI systems T we have the well-known time domain convolution integral representation of the LTI system

$$(Tf)(t) = \int_{-\infty}^{\infty} f(\tau) h_T(t-\tau) \, \mathrm{d}\tau, \quad t \in \mathbb{R}, \tag{1}$$

where $h_T = T(\mathrm{sinc})$ denotes the response of the system T to the sinc function, as well as the frequency domain representation

$$(Tf)(t) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{f}(\omega) \hat{h}_T(\omega) \, \mathrm{e}^{i\omega t} \, \mathrm{d}\omega, \quad t \in \mathbb{R}.$$

A further representation is the following convolution sum sampling representation

$$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(k) h_T(t-k), \quad t \in \mathbb{R}, \tag{2}$$

that uses only the samples of f. For bandlimited input signals f with finite energy, all above expressions are valid, and both the integral in (1) as well as the sum in (2) are absolutely convergent [5]. However, the representations (1) and (2) are often also used for different signal spaces, and it is assumed that they are still valid, at least when the convergence is treated in the sense of distributions.

In this paper we will study the convolution sum (2) for continuous signals that are absolutely integrable and vanish at infinity. In the signal processing literature the convolution sum (2) is usually treated as the limit of the partial sums. In [6] it has been shown that the sequence of partial sums does not necessarily converge if a larger signal space than the space of bandlimited signals with finite energy is considered, and that this result even holds when the convergence is treated in a distributional setting. Hence, the standard approach, taken in the literature, does not work. However, the result in [6] does not prove that there is no other meaningful way to define a generalized convolution sum as a distribution. It might be conceivable that even though the series (2) (understood as the sequence of partial sums) diverges, there exists a different way to define a generalized convolution sum.

We will show that it is not possible to define a meaningful generalized convolution sum for the signal space under consideration. Further, we show that the application of window functions in the convolution sum is useless, because, regardless of the employed window function, we have divergence for certain signals. Therefore, the present paper is a substantial extension of the result in [6]. Additionally, we provide a characterization of certain subspaces for which we have convergence.

II. Notation

Since the problem that we study in this paper is closely related to the one analyzed in [6], parts of the material in the definition and motivation sections are taken from [6].

By c_0 we denote the set of all sequences that vanish at infinity and by ℓ^2 the set of all sequences that are square summable. For $\Omega \subseteq \mathbb{R}$, let $L^p(\Omega)$, $1 \le p < \infty$, be the space of all measurable pth-power Lebesgue integrable functions on Ω, with the usual norm $\|\cdot\|_p$, and $L^\infty(\Omega)$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. $C(\Omega)$, equipped with the supremum norm, is the space of continuous functions on Ω. By $C_0(\mathbb{R})$ we denote the Banach space of all continuous functions on $\mathbb{R}$ that vanish at infinity. The norm is given by $\|f\|_{C_0(\mathbb{R})} = \max_{t\in\mathbb{R}} |f(t)|$. Further, by $C_0^\infty[a,b]$ we denote the space of all infinitely often differentiable functions that are zero outside the interval $[a,b]$. Let $\mathcal{C} = C_0(\mathbb{R}) \cap L^1(\mathbb{R})$. Equipped with the norm $\|f\|_{\mathcal{C}} = \max\{\|f\|_{C_0(\mathbb{R})}, \|f\|_{L^1(\mathbb{R})}\}$, $\mathcal{C}$ becomes a separable Banach space. For all $f \in \mathcal{C}$, we have $\int_{-\infty}^{\infty} |f(t)|^2 \, \mathrm{d}t \le \|f\|_{C_0(\mathbb{R})} \int_{-\infty}^{\infty} |f(t)| \, \mathrm{d}t < \infty$, which shows that $f \in L^2(\mathbb{R})$,

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Holger Boche and Ullrich J. Mönich are with the Technische Universität München, Lehrstuhl für Theoretische Informationstechnik, Germany. e-mail: boche@tum.de, moenich@tum.de. Bernd Meinerzhagen is with the Technische Universität Braunschweig, Institut für Elektronische Bauelemente und Schaltungstechnik, Germany. e-mail: b.meinerzhagen@tu-bs.de.

and consequently $\mathcal{C} \subset L^2(\mathbb{R})$. That is, all $f \in \mathcal{C}$ have finite energy. Let $\hat{f} = \mathcal{F}f$ denote the Fourier transform of a function f, where $\hat{f}$ is to be understood in the distributional sense. According to the Riemann–Lebesgue lemma, we have $\lim_{|t|\to\infty}|\hat{f}(t)| = 0$, i.e., $\hat{f} \in C_0(\mathbb{R})$, for all $f \in \mathcal{C}$, which shows that the Fourier transform of functions in $\mathcal{C}$ has nice properties. Let $\mathcal{C}_c$ denote the set of all functions in $\mathcal{C}$ with compact support. The Bernstein space $\mathcal{B}_\sigma^p$, $\sigma > 0$, $1 \le p \le \infty$, consists of all functions of exponential type at most σ, whose restriction to the real line is in $L^p(\mathbb{R})$ [7, p. 49]. The norm for $\mathcal{B}_\sigma^p$ is given by the L^p-norm on the real line. A function in $\mathcal{B}_\sigma^p$ is called bandlimited to σ. By $\mathcal{PW}_\sigma^p$, $1 \le p \le \infty$, we denote the Paley–Wiener space of functions f with a representation $f(z) = 1/(2\pi)\int_{-\sigma}^{\sigma} g(\omega)\,\mathrm{e}^{iz\omega}\,\mathrm{d}\omega$, $z \in \mathbb{C}$, for some $g \in L^p[-\sigma,\sigma]$. If $f \in \mathcal{PW}_\sigma^p$, then $g(\omega) = \hat{f}(\omega)$. The norm for $\mathcal{PW}_\sigma^p$ is given by $\|f\|_{\mathcal{PW}_\sigma^p} = (1/(2\pi)\int_{-\sigma}^{\sigma}|\hat{f}(\omega)|^p\,\mathrm{d}\omega)^{1/p}$. $\mathcal{PW}_\sigma^2$ is the frequently used space of bandlimited functions with finite energy.

III. Motivation

In the vast majority of textbooks and many publications [8]–[14], the derivation of the sampling theorem and the discussion of aliasing is based on the following representation of the Fourier transform of the cardinal series:

$$\mathcal{F}\left[\sum_{k=-\infty}^{\infty} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}\right] = \mathbf{1}_{[-\pi,\pi]}(\omega)\cdot\sum_{k=-\infty}^{\infty}\hat{f}(\omega-k2\pi). \tag{3}$$

Eq. (3) is typically derived using the Dirac comb, the Poisson formula, and the assumption that a convolution in the time domain always corresponds to a multiplication in the frequency domain, but without providing a rigorous justification for this assumption or a clear specification of the set of functions f for which (3) is valid. For a typical example of such a derivation, we refer to Chapter III in [11]. This method of deriving (3) is so common that it is now addressed as textbook proof [12].

In [6] it was shown that the cardinal series in (3) does typically not converge for $f \in \mathcal{C}$, which implies that (3) is typically not valid for $f \in \mathcal{C}$. This sheds some new light on the validity of many calculations based on (3) aiming at the demonstration of basic phenomena like aliasing and frequency folding. A typical example is the formal calculation of the cardinal series based on (3) for $f(t) = \cos(\omega_0 t)$ with $\pi < \omega_0 < 2\pi$, performed in many textbooks for demonstrating frequency folding (see for instance Example 3.1 in [11]). In this case $\hat{f}$ is not a regular distribution and $f(t)$ does not decay for $t \to \infty$. Therefore, the interpretation of the right hand side of (3) as a tempered distribution and the convergence of the cardinal series become both questionable, and the danger is large that the explanation of a basic and certainly existing phenomenon like frequency folding is based on non-existing mathematical expressions.

Moreover, the cardinal series is often used for obtaining a bandlimited interpolation of f [15], [16], and all results based on the existence of this interpolation become obsolete if the cardinal series diverges. Therefore, it is very important to explore whether the cardinal series may converge for all $f \in \mathcal{C}$ if some generalized method, for instance the summation method of Cesàro, is applied, where instead of

$$\lim_{N\to\infty}\sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)}, \tag{4}$$

the alternative limit

$$\lim_{M\to\infty}\frac{1}{M}\sum_{N=1}^{M}\sum_{k=-N}^{N} f(k)\frac{\sin(\pi(t-k))}{\pi(t-k)} \tag{5}$$

is considered. In the classical paper of Whittaker [17], the sampling theorem was already proved for a much larger set of functions f, when (5) was considered instead of (4). However, the results in the present paper show that there exists no generalized method such that the convergence properties of the cardinal series are changed significantly for $f \in \mathcal{C}$.

IV. Linear Time-Invariant Systems

We briefly review some definitions and facts about stable linear time-invariant (LTI) systems. A linear system T is called time-invariant if $(Tf(\cdot - a))(t) = (Tf)(t-a)$ for all input signals f and all $t, a \in \mathbb{R}$. For each $h_T \in \mathcal{PW}_\pi^\infty$, the convolution integral

$$(Tf)(t) = \int_{-\infty}^{\infty} f(\tau)h_T(t-\tau)\,\mathrm{d}\tau, \quad t \in \mathbb{R}, \tag{6}$$

defines an LTI system on the space $\mathcal{C}$. Since $\mathcal{C} \subset L^1(\mathbb{R})$, this integral is absolutely convergent for all $f \in \mathcal{C}$. Moreover, since $\mathcal{C} \subset L^2(\mathbb{R})$ and $\mathcal{PW}_\pi^\infty \subset \mathcal{PW}_\pi^2$, we can use the frequency domain representation

$$(Tf)(t) = \frac{1}{2\pi}\int_{-\pi}^{\pi}\hat{f}(\omega)\hat{h}_T(\omega)\,\mathrm{e}^{i\omega t}\,\mathrm{d}\omega, \quad t \in \mathbb{R}, \tag{7}$$

and Plancherel's theorem to obtain

$$\begin{aligned}\|Tf\|_{\mathcal{PW}_\pi^2}^2 &= \frac{1}{2\pi}\int_{-\pi}^{\pi}|\hat{f}(\omega)|^2|\hat{h}_T(\omega)|^2\,\mathrm{d}\omega\\ &\le \|h_T\|_{\mathcal{PW}_\pi^\infty}^2\|\hat{f}\|_{L^\infty[-\pi,\pi]}^2\\ &\le \|h_T\|_{\mathcal{PW}_\pi^\infty}^2\|f\|_1^2\\ &\le \|h_T\|_{\mathcal{PW}_\pi^\infty}^2\|f\|_{\mathcal{C}}^2\end{aligned}$$

for all $f \in \mathcal{C}$ and all $h_T \in \mathcal{PW}_\pi^\infty$. This shows that for all $h_T \in \mathcal{PW}_\pi^\infty$, T as defined in (6) is a bounded linear operator from $\mathcal{C}$ into $\mathcal{PW}_\pi^2$, which implies that T is stable. In this paper we only consider LTI systems with $h_T \in \mathcal{PW}_\pi^\infty$.

A typical example of such an LTI system is a bandpass filter, i.e., a system with $\hat{h}_T(\omega) = \mathbf{1}_{[\omega_1,\omega_2]}(\omega)$, $\omega \in [-\pi,\pi]$, where $-\pi \le \omega_1 < \omega_2 \le \pi$. $\mathbf{1}_A$ denotes the indicator function of the set A.

V. Distributions

Distributions are continuous linear functionals on a space of test functions. Two common test functions spaces are $\mathcal{D}$ and $\mathcal{S}$. $\mathcal{D}$ is the space of all functions $\phi : \mathbb{R} \to \mathbb{C}$ that have continuous derivatives of all orders and are zero outside some finite interval. $\mathcal{D}'$ denotes the dual space of $\mathcal{D}$, i.e., the space of all distributions that can be defined on $\mathcal{D}$. The Schwartz space $\mathcal{S}$ consists of all continuous functions

$\phi : \mathbb{R} \to \mathbb{C}$ that have continuous derivatives of all orders and fulfill $\sup_{t\in\mathbb{R}} |t^a \phi^{(b)}(t)| < \infty$ for all $a, b \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$. $p_{a,b}(\phi) = \sup_{t\in\mathbb{R}} |t^a \phi^{(b)}(t)|$ defines a family of seminorms. A sequence $\{\phi_k\}_{k\in\mathbb{N}} \subset \mathcal{S}$ is said to converge in $\mathcal{S}$ if there exists an element $\phi \in \mathcal{S}$ such that $\lim_{k\to\infty} p_{a,b}(\phi_k - \phi) = 0$ for all $a, b \in \mathbb{N}_0$. $\mathcal{S}'$ denotes the dual space of $\mathcal{S}$. A nice property of $\mathcal{S}$ is that the Fourier transform maps $\mathcal{S}$ onto itself, i.e., we have $\mathcal{F}\mathcal{S} = \mathcal{S}$. Clearly, we have $\|\phi\|_\infty < \infty$ and $\|\phi\|_1 < \infty$ for all $\phi \in \mathcal{S}$. In contrast to $\mathcal{S}$, the Fourier transform of functions in $\mathcal{D}$ is not necessarily again in $\mathcal{D}$. Hence, it is problematic to define the Fourier transform for $\mathcal{D}'$.

Typical examples of distributions in $\mathcal{S}'$ are the Dirac delta function δ, the derivatives of the Dirac delta function $\delta^{(l)}$, $l \geq 1$, all finite linear combinations of distributions, and the Dirac comb $Ш(t) = \sum_{k=-\infty}^{\infty} \delta(t-k)$.

For locally integrable functions g we can define the linear functional

$$\phi \mapsto \int_{-\infty}^{\infty} g(t)\phi(t)\ \mathrm{d}t \tag{8}$$

on the space $\mathcal{D}$. It can be proven that this functional is continuous and thus defines a distribution [18]. If g further fulfills $\int_{-\infty}^{\infty} |g(t)|(1+|t|)^{-m}\ \mathrm{d}t < \infty$ for some $m \geq 0$, then (8) defines a continuous linear functional on $\mathcal{S}$. Distributions of this type are called regular distributions.

For a functional $f \in \mathcal{S}'$, we denote by $\langle f, \phi\rangle = f(\phi)$ the number that f assigns to the test function $f \in \mathcal{S}$. A sequence of distributions $\{f_n\}_{n\in\mathbb{N}}$ in $\mathcal{S}'$ is said to converge in $\mathcal{S}'$, if for every $\phi \in \mathcal{S}$ the sequence of numbers $\{\langle f_n, \phi\rangle\}_{n\in\mathbb{N}}$ converges. Thus, a sequence of regular distributions, which is induced by a sequence of functions $\{g_n\}_{n\in\mathbb{N}}$ according to (8), converges in $\mathcal{S}'$, if for every $\phi \in \mathcal{S}$ the sequence of numbers $\{\int_{-\infty}^{\infty} g_n(t)\phi(t)\ \mathrm{d}t\}_{n\in\mathbb{N}}$ converges. For further details about distributions, we refer the reader to [18].

VI. Convolution Sum System Representation

In many signal processing books and publications, sampling of a continuous signal is often modeled as multiplication with a Dirac comb [2], [3], [15], [19]:

$$\begin{aligned} f_Ш(t) &= f(t) \cdot Ш(t) \\ &= f(t) \cdot \sum_{k=-\infty}^{\infty} \delta(t-k) \\ &= \sum_{k=-\infty}^{\infty} f(k)\delta(t-k). \end{aligned} \tag{9}$$

For $f \in \mathcal{C}$, all expressions in (9) are well-defined distributions in $\mathcal{S}'$, because f is continuous and decays. The convergence of the series in (9) needs to be treated in the sense of distributions.

Formally, the following manipulation

$$\text{“}(Tf)(t) = (f_Ш * h_T)(t) = \sum_{k=-\infty}^{\infty} f(k)h_T(t-k)\text{”}$$

is often performed to obtain the output Tf of a stable LTI system T, and it is assumed that these expressions are meaningful, at least in a distributional setting. However, this is not at all clear, and in the signal processing literature, starting from classical books like [10], [15], [20], the question has not been properly treated.

For $f \in \mathcal{PW}_\pi^2$ we have, in addition to the convolution integral representation (6) and the frequency domain representation (7), the following convolution sum sampling representation

$$(Tf)(t) = \sum_{k=-\infty}^{\infty} f(k)h_T(t-k), \quad t \in \mathbb{R}, \tag{10}$$

where the series in (10) converges uniformly on the whole real axis. However, the infinite sum in (10) is often also used for other signal spaces. Then the convergence is not guaranteed and has to be checked from case to case. In [21] convolution sum system representations were analyzed for signals in $\mathcal{PW}_\pi^1$ and non-equidistant sampling patterns, and it was proved that for every sampling pattern that is a complete interpolating sequence and all $t \in \mathbb{R}$, there exists a stable LTI system T and a signal $f \in \mathcal{PW}_\pi^1$, such that the corresponding convolution sum approximation process diverges at t.

The distributional behavior of

$$\sum_{k=-\infty}^{\infty} f(k)h_T(t-k), \quad t \in \mathbb{R}, \tag{11}$$

was analyzed with respect to the convergence of the sequence of partial sums in [22] for signals $f \in \mathcal{PW}_\pi^1$, and in [6] for signals $f \in \mathcal{C}$.

In this paper we consider signals $f \in \mathcal{C}$ and stable LTI systems with $h_T \in \mathcal{PW}_\pi^\infty$. Let

$$(T_N f)(t) = \sum_{k=-N}^{N} f(k)h_T(t-k), \quad t \in \mathbb{R}, \tag{12}$$

denote the N-th partial sum. In order to analyze the distributional convergence behavior of (11), the expression

$$\langle T_N f, \phi\rangle$$

has to be analyzed for all test functions ϕ as N tends to infinity. We have

$$\begin{aligned} \langle T_N f, \phi\rangle &= \int_{-\infty}^{\infty} \left(\sum_{k=-N}^{N} f(k)h_T(t-k) \right) \phi(t)\ \mathrm{d}t \\ &= \sum_{k=-N}^{N} f(k) \int_{-\infty}^{\infty} h_T(t-k)\phi(t)\ \mathrm{d}t \\ &= \sum_{k=-N}^{N} f(k)c_k(h_T, \phi), \end{aligned}$$

where we introduced the abbreviation

$$c_k(h_T, \phi) = \int_{-\infty}^{\infty} h_T(t-k)\phi(t)\ \mathrm{d}t.$$

Since

$$\sum_{k=-\infty}^{\infty} |f(k)c_k(h_T,\phi)| \leq \|f\|_{\mathcal{C}} \sum_{k=-\infty}^{\infty} |c_k(h_T,\phi)|,$$

we see that

$$\sum_{k=-\infty}^{\infty} f(k)c_k(h_T, \phi),$$

or, equivalently, $\{\langle T_N f, \phi\rangle\}_{N\in\mathbb{N}}$ converges for all $\phi \in \mathcal{S}$ if

$$\sum_{k=-\infty}^{\infty} |c_k(h_T, \phi)| < \infty$$

for all $\phi \in \mathcal{S}$. The above short calculation, which was originally given in [6], shows that

$$\sum_{k=-\infty}^{\infty} |c_k(h_T, \phi)|$$

is an important quantity in the analysis of the distributional convergence behavior of the convolution sum. We will encounter this expression again in Lemma 1.

The convergence of (11) for $f \in \mathcal{C}$ and $h_T \in \mathcal{PW}_\pi^\infty$ was analyzed in [6], and it was shown that there exist signals and systems such that (11), understood as the sequence of partial sums, diverges in $\mathcal{S}'$. More precisely, it was proved that (11) converges in $\mathcal{S}'$ for all $f \in \mathbb{C}$ if and only if $h_T \in \mathcal{B}_\pi^1$. As an example, the ideal low-pass filter and the Hilbert transform are two stable LTI systems that do not satisfy this condition and hence for these systems there exists a signal $f \in \mathcal{C}$ such that (11) diverges in $\mathcal{S}'$.

The result in [6] shows that the expression (11) cannot be used to define a convolution sum that is meaningful for all $f \in \mathcal{C}$ and all $h_T \in \mathcal{PW}_\pi^\infty$ in a distributional setting. However, there might exist other approaches, apart from (11), to define a generalized convolution sum, for example windowed convolutions sums.

In this paper we go one step further and ask whether it is at all possible to meaningfully define a generalized convolution sum as a distribution in $\mathcal{S}'$. The answer, as we will prove in Section VII, is “no”. This result implies that there exists no windowing procedure that leads to a convergent convolution sum. We will discuss window functions in more detail in Section VII.

Interestingly, the convolution integral system representation (6) and the frequency domain representation are valid for all $f \in \mathcal{C}$ and all $h_T \in \mathcal{PW}_\pi^\infty$, i.e., do not have convergence problems like the convolution sum. This shows that there is a significant difference in the convergence behavior.

VII. Non-Existence of a Generalized Convolution Sum System Representation

The main objective of this paper is to prove that it is not possible to define a generalized convolution sum for all $f \in \mathcal{C}$ and all $h_T \in \mathcal{PW}_\pi^\infty$ as an element of $\mathcal{S}'$ that satisfies two reasonable assumptions.

The two assumptions are discussed next. By $\Psi(f, h_T) \in \mathcal{S}'$ we denote the hypothetical convolution of f and h_T. The first assumption is that for all $f \in \mathcal{C}_c$, $\Psi(f, h_T)$ is a regular distribution that is generated by the classical convolution sum

$$\sum_{k=-\infty}^{\infty} f(k) h_T(t-k), \quad t \in \mathbb{R}. \tag{13}$$

In other words, for all “nice” signals in $\mathcal{C}$, $\Psi(f, h_T)$ should coincide with the classical convolution sum:

(A1) For all $h_T \in \mathcal{PW}_\pi^\infty$ and all $f \in \mathcal{C}_c$ we have

$$\langle \Psi(f, h_T), \phi\rangle = \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} f(k) h_T(t-k)\phi(t)\,\mathrm{d}t$$

for all $\phi \in \mathcal{S}$.

Clearly, for $f \in \mathcal{C}_c$, the convolution sum (13) has only finitely many summands. Hence, there exist no convergence problems, and (13), being a finite linear combination of $\mathcal{PW}_\pi^\infty$ signals, is a signal in $\mathcal{PW}_\pi^\infty$.

Using the axiom of choice it is possible to construct linear functionals on Banach spaces that have a very complicated structure. However, it is important that $\Psi(f, h_T)$ has good properties for general f also, i.e., $f \notin \mathcal{C}_c$. In particular, one also needs a practical way to compute $\Psi(f, h_T)$. This leads us to our second assumption:

(A2) For all $h_T \in \mathcal{PW}_\pi^\infty$ and all $f \in \mathcal{C}$ there exists a sequence $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}_c$, with

$$\lim_{n\to\infty} \|f - f_n\|_{\mathcal{C}} = 0$$

such that

$$\lim_{n\to\infty} \langle \Psi(f_n, h_T), \phi\rangle = \langle \Psi(f, h_T), \phi\rangle$$

for all $\phi \in \mathcal{S}$.

Hence, we require that for each signal $f \in \mathcal{C}$ we have a method to approximate $\Psi(f, h_T)$ by a sequence of simple, regular distributions that have a representation as a finite convolution sum.

Remark 1. Assumption (A2) is clearly satisfied if $\Psi(f, h_T)$ is continuous in the first argument, i.e., if for all $f \in \mathcal{C}$ and all sequences $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$ with

$$\lim_{n\to\infty} \|f - f_n\|_{\mathcal{C}} = 0$$

we have

$$\lim_{n\to\infty} \langle \Psi(f_n, h_T), \phi\rangle = \langle \Psi(f, h_T), \phi\rangle$$

for all $\phi \in \mathcal{S}$.

Continuity with respect to f can be interpreted as a robustness guarantee. Small changes in f should only produce small changes in the obtained generalized convolution.

Now we state our main theorem.

Theorem 1. *There exists no mapping $\Psi\colon \mathcal{C} \times \mathcal{PW}_\pi^\infty \to \mathcal{S}'$ that satisfies conditions (A1) and (A2).*

For the proof of Theorem 1 we need the following lemma, the proof of which is postponed until Section X.

Lemma 1. *There exist functions $f^* \in \mathcal{C}$, $h_T^* \in \mathcal{PW}_\pi^\infty$, and $\phi^* \in \mathcal{S}$, as well as a sequence $\{\phi_n^*\}_{n\in\mathbb{N}} \subset \mathcal{S}$, such that*

1) $\sum_{k=-\infty}^{\infty} |c_k(h_T^*, \phi_n^*)| < \infty$ *for all* $n \in \mathbb{N}$,
2) $\lim_{n\to\infty} \phi_n^* = \phi^*$ *in* $\mathcal{S}$,
3) $\limsup_{n\to\infty} |\sum_{k=-\infty}^{\infty} f^*(k) c_k(h_T^*, \phi_n^*)| = \infty$.

In the proof of Lemma 1 we will see that for the functions $h_T^* \in \mathcal{PW}_\pi^\infty$ and $\phi^* \in \mathcal{S}$, we have

$$\lim_{N\to\infty} \sum_{k=-N}^{N} |c_k(h_T^*, \phi^*)| = \infty.$$

In Section VIII we will further refine this statement.

Next, we present the proof of Theorem 1.

Proof of Theorem 1. We prove the theorem by contradiction. Assume that there exists a mapping $\Psi\colon \mathcal{C}\times\mathcal{PW}_\pi^\infty \to \mathcal{S}'$ that satisfies conditions (A1) and (A2). According to Lemma 1, there exist functions $f^* \in \mathcal{C}$, $h_T^* \in \mathcal{PW}_\pi^\infty$, and $\phi^* \in \mathcal{S}$, as well as sequence $\{\phi_n^*\}_{n\in\mathbb{N}} \subset \mathcal{S}$, such that

$$\sum_{k=-\infty}^{\infty} |c_k(h_T^*, \phi_n^*)| < \infty$$

for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \phi_n^* = \phi^* \text{ in } \mathcal{S},$$

and

$$\limsup_{n\to\infty} \left| \sum_{k=-\infty}^{\infty} f^*(k) c_k(h_T^*, \phi_n^*) \right| = \infty. \tag{14}$$

Let $\mathcal{K} \subset \mathcal{S}$ denote the set of all functions $\phi \in \mathcal{S}$ for which

$$\sum_{k=-\infty}^{\infty} |c_k(h_T^*, \phi)| < \infty,$$

and define

$$T_\phi f = \sum_{k=-\infty}^{\infty} f(k) c_k(h_T^*, \phi).$$

For $f \in \mathcal{C}$ and $\phi \in \mathcal{K}$ we have

$$|T_\phi f| \le \|f\|_{\mathcal{C}} \sum_{k=-\infty}^{\infty} |c_k(h_T^*, \phi)| = \|f\|_{\mathcal{C}} C_1(\phi).$$

That is, for $\phi \in \mathcal{K}$, $T_\phi\colon \mathcal{C} \to \mathbb{C}$ is a continuous linear operator. According to assumption (A2), there exists a sequence $\{f_n^*\}_{n\in\mathbb{N}} \subset \mathcal{C}_{\mathrm{c}}$ with

$$\lim_{n\to\infty} \|f^* - f_n^*\|_{\mathcal{C}} = 0$$

and

$$\lim_{n\to\infty} \langle \Psi(f_n^*, h_T^*), \phi\rangle = \langle \Psi(f^*, h_T^*), \phi\rangle \tag{15}$$

for all $\phi \in \mathcal{K}$. Further, from assumption (A1) it follows that

$$T_\phi f_n^* = \langle \Psi(f_n^*, h_T^*), \phi\rangle \tag{16}$$

for all $\phi \in \mathcal{K}$ and all $n \in \mathbb{N}$. Since T_ϕ is continuous, we have

$$\lim_{n\to\infty} T_\phi f_n^* = T_\phi f^*$$

for all $\phi \in \mathcal{K}$. Using (15) and (16), we see that

$$T_\phi f^* = \langle \Psi(f^*, h_T^*), \phi\rangle \tag{17}$$

for all $\phi \in \mathcal{K}$. Since $\Psi(f^*, h_T^*)$ is, by assumption, a continuous linear functional on $\mathcal{S}$, we have

$$\begin{aligned}\langle \Psi(f^*, h_T^*), \phi^*\rangle &= \lim_{n\to\infty} \langle \Psi(f^*, h_T^*), \phi_n^*\rangle \\ &= \lim_{n\to\infty} T_{\phi_n^*} f^*,\end{aligned}$$

where the last equality follows from (17). Thus, there exists an $n_0 \in \mathbb{N}$ such that

$$|T_{\phi_n^*} f^*| \le |\langle \Psi(f^*, h_T^*), \phi^*\rangle| + 1$$

for all $n \ge n_0$. However, from (14) we see that

$$\limsup_{n\to\infty} |T_{\phi_n^*} f^*| = \infty,$$

which is a contradiction. □

Theorem 1 can be rephrased as follows.

Corollary 1. *A generalized convolution sum that satisfies conditions (A1) and (A2) cannot be defined as a distribution.*

We conclude this section with a discussion about Theorem 1 and window functions.

Window functions play an important role in signal processing because many problems benefit from a proper choice of a window function [2], [17]. It is frequently observed that other window functions than the rectangular window lead to a better approximation behavior. For this reason, several window functions, e.g., triangular windows, trapezoidal windows, and cosine roll-off windows, have been proposed.

We can modify the convolution sum (11) by adding a window function $w_k(N)$. A window function has the property that for each $N \in \mathbb{N}$ there exists a $M_N \in \mathbb{N}$ such that $w_k(N) = 0$ for all $|k| > M_N$. This results in the expression

$$\lim_{N\to\infty} \sum_{k=-M_N}^{M_N} w_k(N) f(k) h_T(t-k). \tag{18}$$

Then the result in [6] about the divergence of (11) in $\mathcal{S}'$ is equivalent to the statement that the rectangular window

$$w_k^{\text{rect}}(N) = \begin{cases} 1, & |k| \le N, \\ 0, & |k| > N, \end{cases}$$

does not lead to an approximation process that converges in $\mathcal{S}'$. However, [6] makes no statement about the convergence of (18) if other window functions, such as the triangular window

$$w_k^{\text{tri}}(N) = \begin{cases} 1 - |k|/N, & |k| \le N, \\ 0, & |k| > N, \end{cases}$$

are used. Window functions $w_k(N)$ have the property $\lim_{N\to\infty} w_k(N) = 1$ for all $k \in \mathbb{Z}$.

Before we discuss the implications of Theorem 1 on the convergence of the convolution sum with window function, we present two prominent examples which benefit from windowing: 1) Fourier series for the approximation of continuous signals and 2) certain generalizations of the Shannon sampling series for classes of bandlimited signals.

We start with the Fourier series example. For continuous 2π-periodic signals f, let

$$(S_N f)(t) = \sum_{k=-N}^{N} c_k(f)\, \mathrm{e}^{i\omega k} \tag{19}$$

denote the N-th partial sum of the Fourier series, where

$$c_k(f) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t)\, \mathrm{e}^{-i\omega k}\, \mathrm{d}t$$

are the Fourier coefficients. A classical result by Fejér states that $(S_N f)(t)$ does not necessarily converge to $f(t)$. More specifically, there exists a signal $f_1 \in C([-\pi, \pi])$ such that

$$\limsup_{N\to\infty} |f_1(t) - (S_N f_1)(t)| = \infty$$

for infinitely many $t \in [-\pi, \pi]$. The expression in (19) corresponds exactly to the rectangular windowed Fourier series. On the other hand, Fejér showed that

$$(F_N f)(t) = \sum_{k=-N}^{N} w_k^{\text{tri}}(N) c_k(f)\, \mathrm{e}^{i\omega k} \tag{20}$$

converges uniformly to f for all $f \in C([-\pi, \pi])$. The expression in (20) corresponds to a triangular windowing of the Fourier series. The same result holds if a trapezoidal or cosine roll-off window function is employed. Thus, this example shows that by the choice of a suitable window function, a stable approximation can be achieved with the windowed Fourier transform.

A similar behavior occurs for the Shannon sampling series. Originally, Claude Shannon considered the sampling series only for signals in $\mathcal{PW}_\pi^2$, but subsequently applications in signal processing, e.g., stochastic processes and Wiener filtering, made it necessary to develop a sampling theorem for larger signal classes. As an example, we discuss the signal space $\mathcal{PW}_\pi^1$. There exists a signal $f_1 \in \mathcal{PW}_\pi^1$ such that

$$\lim_{N\to\infty} \max_{t\in\mathbb{R}} \left| f_1(t) - \sum_{k=-N}^{N} f_1(k) \frac{\sin(\pi(t-k))}{\pi(t-k)} \right| = \infty,$$

i.e., rectangular windowing leads to strong divergence of the peak value of the Shannon sampling series for certain signals in $\mathcal{PW}_\pi^1$ [23], [24]. On the other hand, using the result from [17], it can be easily shown that the sampling series with triangular windowing

$$\sum_{k=-N}^{N} w_k^{\text{tri}}(N) f(k) \frac{\sin(\pi(t-k))}{\pi(t-k)}$$

converges globally uniformly for all $f \in \mathcal{PW}_\pi^1$. The same results holds for trapezoidal windows. This shows that the suitable choice of a window function leads to a stable approximation of $\mathcal{PW}_\pi^1$ signals by sampling series.

Now we come to a corollary of Theorem 1, which shows that no window function can create a convolution sum that always converges in $\mathcal{S}'$.

Corollary 2. *There exists no window function $w_k(N)$ such that*

$$\sum_{k=-M_N}^{M_N} w_k(N) f(k) h_T(t-k)$$

converges in $\mathcal{S}'$ for all $f \in \mathcal{C}$ and all $h_T \in \mathcal{PW}_\pi^\infty$ as N tends to infinity.

Corollary 2 highlights that a behavior like the one in the last two examples cannot occur for the convolution sum system representation because we have divergence regardless of the windowing function.

Proof of Corollary 2. We use an indirect proof. Assume that there exists a window function $w_k(N)$ such that

$$\lim_{N\to\infty} \sum_{k=-M_N}^{M_N} w_k(N) f(k) h_T(t-k)$$

converges in $\mathcal{S}'$ for all $f \in \mathcal{C}$ and all $h_T \in \mathcal{PW}_\pi^\infty$. Then

$$\begin{aligned}\langle \Psi(f, h_T), \phi\rangle \\ &= \lim_{N\to\infty} \int_{-\infty}^{\infty} \sum_{k=-M_N}^{M_N} w_k(N) f(k) h_T(t-k) \phi(t)\, \mathrm{d}t\end{aligned}$$

defines a distribution $\Psi(f, h_T) \in \mathcal{S}'$. We prove that $\Psi(f, h_T)$ satisfies the assumptions (A1) and (A2), and thus create a contradiction to Theorem 1.

Let $h_T \in \mathcal{PW}_\pi^\infty$ and $f_c \in \mathcal{C}_\mathrm{c}$ be arbitrary but fixed, and let L be the smallest natural number such that $f_c(k) = 0$ for all $|k| \geq L$. We have

$$\begin{aligned}\langle \Psi(f_c, h_T), \phi\rangle \\ &= \lim_{N\to\infty} \int_{-\infty}^{\infty} \sum_{k=-M_N}^{M_N} w_k(N) f_c(k) h_T(t-k) \phi(t)\, \mathrm{d}t \\ &= \lim_{N\to\infty} \sum_{k=-L}^{L} w_k(N) f_c(k) \int_{-\infty}^{\infty} h_T(t-k)\phi(t)\, \mathrm{d}t \\ &= \sum_{k=-L}^{L} f_c(k) \int_{-\infty}^{\infty} h_T(t-k)\phi(t)\, \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \sum_{k=-L}^{L} f_c(k) h_T(t-k)\phi(t)\, \mathrm{d}t\end{aligned}$$

for all $\phi \in \mathcal{S}$. Hence, assumption (A1) is satisfied.

Next, we prove that assumption (A2) is also satisfied. Let $h_T \in \mathcal{PW}_\pi^\infty$ and $\phi \in \mathcal{S}$ be arbitrary but fixed. Further, for $N \in \mathbb{N}$, let $\Gamma_N \colon \mathcal{C} \to \mathbb{C}$ be defined by

$$\Gamma_N f = \int_{-\infty}^{\infty} \sum_{k=-M_N}^{M_N} w_k(N) f(k) h_T(t-k)\phi(t)\, \mathrm{d}t.$$

This gives us a sequence of continuous linear functionals $\{\Gamma_N\}_{N\in\mathbb{N}}$. According to our assumption, $\{\Gamma_N f\}_{N\in\mathbb{N}}$ converges for all $f \in \mathcal{C}$. Hence, as a consequence of the Banach–Steinhaus theorem [25, p. 45, Theorem 2.7], the linear functional $\overline{\Gamma} \colon \mathcal{C} \to \mathbb{C}$, defined by

$$\overline{\Gamma} f = \lim_{N\to\infty} \Gamma_N f = \langle \Psi(f, h_T), \phi\rangle$$

is continuous as well. The continuity of $\langle \Psi(f, h_T), \phi\rangle$ with respect to f implies that the statement of assumption (A2) is true. □

VIII. Size of the Divergence Set

In this section we study the size of certain sets of functions and systems in terms of Baire categories, the basic definitions of which are reviewed next. A subset $\mathcal{M}$ of a Banach space $\mathcal{X}$ is said to be nowhere dense in $\mathcal{X}$ if the interior of the closure of $\mathcal{M}$ is empty. $\mathcal{M}$ is said to be of the first category (or meager) if $\mathcal{M}$ is the countable union of sets each of which is nowhere dense in $\mathcal{X}$. $\mathcal{M}$ is said to be of the second category (or nonmeager) if it is not of the first category. The complement of a set of the first category is called a residual set. Topologically, sets of the first category may be considered "small". Accordingly, residual sets, being the complements of sets of the first category, can be considered "large". In a complete metric space, any residual set is dense and a set of the second category, due to Baire's theorem [26].

For $h_T^* \in \mathcal{PW}_\pi^\infty$ and the sequence $\{\phi_n^*\}_{n\in\mathbb{N}} \subset \mathcal{S}$ from Lemma 1, the expression

$$G(f, h_T^*, \phi_n^*) = \sum_{k=-\infty}^{\infty} f(k) c_k(h_T^*, \phi_n^*)$$

is well-defined and finite for all $n \in \mathbb{N}$. Further, for each $n \in \mathbb{N}$, $G(f, h_T^*, \phi_n^*)$ defines a continuous linear functional with respect to f, because

$$|G(f, h_T^*, \phi_n^*)| \le \sum_{k=-\infty}^{\infty} |f(k) c_k(h_T^*, \phi_n^*)| \le \|f\|_{\mathcal{C}} \sum_{k=-\infty}^{\infty} |c_k(h_T^*, \phi_n^*)|.$$

Let

$$D_1 = \left\{ f \in \mathcal{C} \colon \limsup_{n\to\infty} |G(f, h_T^*, \phi_n^*)| = \infty \right\}.$$

For all $f \in D_1$, the convolution sum

$$\sum_{k=-\infty}^{\infty} f(k) h_T^*(t-k), \quad t \in \mathbb{R},$$

diverges in $\mathcal{S}'$, i.e., we have

$$\lim_{N\to\infty} \int_{-\infty}^{\infty} \sum_{k=-N}^{N} f(k) h_T^*(t-k) \phi(t) \, \mathrm{d}t = \infty$$

for some $\phi \in \mathcal{S}$. The next theorem shows that the set D_1 is big in a topological sense.

Theorem 2. *Let $h_T^* \in \mathcal{PW}_\pi^\infty$ and $\{\phi_n^*\}_{n\in\mathbb{N}}$ be as in the definition of Lemma 1. The set D_1 is a residual set in $\mathcal{C}$.*

The proof of Theorem 2 will be given in Section X.

As we have discussed in Section VI and have seen in Lemma 1, the systems $h_T \in \mathcal{PW}_\pi^\infty$ for which there exists a $\phi \in \mathcal{S}$ such that

$$\lim_{N\to\infty} \sum_{k=-N}^{N} |c_k(h_T, \phi)| = \infty,$$

can be problematic. Hence, it is interesting to know the size of the set of systems for which this occurs. Let

$$D_2 = \left\{ h_T \in \mathcal{PW}_\pi^\infty \colon \text{ there exists a } \phi \in \mathcal{S} \text{ with } \lim_{N\to\infty} \sum_{k=-N}^{N} |c_k(h_T, \phi)| = \infty \right\}.$$

Again, this set is big in a topological sense.

Theorem 3. *D_2 is a residual set in $\mathcal{PW}_\pi^\infty$.*

The proof of Theorem 3 will be given in Section X.

Remark 2. We actually will prove more. We will show that the set of $\phi \in \mathcal{S}$, for which

$$D_2(\phi) = \left\{ h_T \in \mathcal{PW}_\pi^\infty \colon \lim_{N\to\infty} \sum_{k=-N}^{N} |c_k(h_T, \phi)| = \infty \right\}$$

is a residual set, is non-empty.

IX. Discussion

A. Modification of Assumption (A2)

In order to prove that the convolution sum (13) cannot be reasonably defined even in a distributional setting for all $f \in \mathcal{C}$ and all $h_T \in \mathcal{PW}_\pi^\infty$, we made two assumptions. Assumption (A1) is very natural and cannot be weakened from a practical point of view. We require that the convolution, as an element of $\mathcal{S}'$, always coincides with the ordinary convolution sum (13) for all nice signals $f \in \mathcal{C}_\mathrm{c}$. However, assumption (A2) can be further analyzed and probably replaced by other assumptions. We will discuss one alternative choice next.

(A2') For all $h_T \in \mathcal{PW}_\pi^\infty$, all $f \in \mathcal{C}$, all sequences $\{f_n\}_{n\in\mathbb{N}} \subset \mathcal{C}$, with

$$\lim_{n\to\infty} \|f - f_n\|_{\mathcal{C}} = 0,$$

and all $\phi \in \mathcal{S}$ for which

$$\{\langle \Psi(f_n, h_T), \phi \rangle\}_{n\in\mathbb{N}}$$

converges, we have

$$\lim_{n\to\infty} \langle \Psi(f_n, h_T), \phi \rangle = \langle \Psi(f, h_T), \phi \rangle.$$

Note that assumption (A2) does not imply assumption (A2'), nor does assumption (A2') imply assumption (A2).

Replacing (A2) by (A2'), we obtain the same result as in Theorem 1.

Theorem 4. *There exists no mapping $\Psi \colon \mathcal{C} \times \mathcal{PW}_\pi^\infty \to \mathcal{S}'$ that satisfies conditions (A1) and (A2').*

Proof. We prove the theorem by contradiction. Assume that there exists a mapping $\Psi \colon \mathcal{C} \times \mathcal{PW}_\pi^\infty \to \mathcal{S}'$ that satisfies conditions (A1) and (A2'). According to Lemma 1, there exist functions $f^* \in \mathcal{C}$, $h_T^* \in \mathcal{PW}_\pi^\infty$, and $\phi^* \in \mathcal{S}$, as well as sequence $\{\phi_n^*\}_{n\in\mathbb{N}} \subset \mathcal{S}$, such that

$$\sum_{k=-\infty}^{\infty} |c_k(h_T^*, \phi_n^*)| < \infty$$

for all $n \in \mathbb{N}$,

$$\lim_{n\to\infty} \phi_n^* = \phi^* \text{ in } \mathcal{S},$$

and

$$\limsup_{n\to\infty} \left| \sum_{k=-\infty}^{\infty} f^*(k) c_k(h_T^*, \phi_n^*) \right| = \infty. \tag{21}$$

Let $\mathcal{K} \subset \mathcal{S}$ denote the set of all functions $\phi \in \mathcal{S}$ for which

$$\sum_{k=-\infty}^{\infty} |c_k(h_T^*, \phi)| < \infty,$$

and define

$$T_\phi f = \sum_{k=-\infty}^{\infty} f(k) c_k(h_T^*, \phi).$$

From the proof of Theorem 1, we already know that for $\phi \in \mathcal{K}$, $T_\phi \colon \mathcal{C} \to \mathbb{C}$ is a continuous linear operator. Let $\{f_n^*\}_{n\in\mathbb{N}} \subset \mathcal{C}_\mathrm{c}$ be a sequence of functions with

$$\lim_{n\to\infty} \|f^* - f_n^*\|_{\mathcal{C}} = 0.$$

Such a sequence is easily constructed. Take for example

$$f_n^*(t) = f^*(t)\zeta\left(\frac{t}{n}\right),$$

where ζ is an arbitrary function in $C_0^\infty[-1,1]$ with $\zeta(0) = 1$. It follows that for all $\phi \in \mathcal{K}$, $\{T_\phi f_n^*\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$. According to assumption (A1), we have

$$T_\phi f_n^* = \langle \Psi(f_n^*, h_T^*), \phi\rangle$$

for all $\phi \in \mathcal{S}$ and all $n \in \mathbb{N}$. Hence, it follows that the limit

$$\lim_{n\to\infty} \langle \Psi(f_n^*, h_T^*), \phi\rangle$$

exists for all $\phi \in \mathcal{K}$, and assumption (A2') implies that

$$\lim_{n\to\infty} \langle \Psi(f_n^*, h_T^*), \phi\rangle = \langle \Psi(f^*, h_T^*), \phi\rangle$$

for all $\phi \in \mathcal{K}$. Hence, we obtain

$$\begin{aligned}\langle \Psi(f^*, h_T^*), \phi\rangle &= \lim_{n\to\infty} T_\phi f_n^* \\ &= T_\phi f^* \end{aligned} \tag{22}$$

for all $\phi \in \mathcal{K}$. Since $\Psi(f^*, h_T^*)$ is, by assumption, a continuous linear functional on $\mathcal{S}$, we have

$$\begin{aligned}\langle \Psi(f^*, h_T^*), \phi^*\rangle &= \lim_{n\to\infty} \langle \Psi(f^*, h_T^*), \phi_n^*\rangle \\ &= \lim_{n\to\infty} T_{\phi_n^*} f^*,\end{aligned}$$

where the last equality follows from (22). Thus, there exists an $n_0 \in \mathbb{N}$ such that

$$|T_{\phi_n^*} f^*| \leq |\langle \Psi(f^*, h_T^*), \phi^*\rangle| + 1$$

for all $n \geq n_0$. However, from (21) we see that

$$\limsup_{n\to\infty} |T_{\phi_n^*} f^*| = \infty,$$

which is a contradiction. □

B. Convergence for Certain Signal Spaces

The finite convolution sum

$$\sum_{k=-N}^{N} f(k) h_T(t-k) \tag{23}$$

has two inputs, the signal f and the system response h_T, and hence can mathematically be seen as a bilinear operator. A key question to ask is: For what inputs f and h_T does (23) converge as N tends to infinity? In [6] it has been shown that $h_T \in \mathcal{B}_\pi^1$ is a sufficient and necessary condition for the convergence of (23) in $\mathcal{S}'$ for all $f \in \mathcal{C}$.

It is also possible to study which condition we have to impose on the input signal space $\mathcal{I}$ in order that (23) converges in $\mathcal{S}'$ for all $h_T \in \mathcal{PW}_\pi^\infty$ and all $f \in \mathcal{I}$. Clearly, a necessary condition for (23) to be well-defined is that the sequence of samples $\{f(k)\}_{k\in\mathbb{Z}}$ is well-defined. This is, for example, the case if all signals in $\mathcal{I}$ are continuous.

In the following, we consider continuously embedded closed subspaces of $C(\mathbb{R})$ as input signal space $\mathcal{I}$. Let $\|\cdot\|_\mathcal{I}$ denote the norm of the Banach space $\mathcal{I}$. Since we require that $\mathcal{I}$ is continuously embedded in $C(\mathbb{R})$, there exists a constant C_2 such that $\|f\|_\infty \leq C_2\|f\|_\mathcal{I}$ for all $f \in \mathcal{I}$. This assumption ensures that the point evaluation functionals $f \mapsto f(k)$, $k \in \mathbb{Z}$, are bounded, and hence, from a practical point of view, is no restriction.

Note that all signals in $\mathcal{C}$ are continuous, and $\mathcal{C}$ is continuously embedded in $C(\mathbb{R})$. However, as we have seen, the space $\mathcal{C}$ is too large because there exist $f \in \mathcal{C}$ and $h_T \in \mathcal{PW}_\pi^\infty$ such that (23) diverges in $\mathcal{S}'$.

Next, we present an input signal space $\mathcal{I}$ for which we have convergence for all $h_T \in \mathcal{PW}_\pi^\infty$ and all $f \in \mathcal{I}$.

Theorem 5. *Let $\mathcal{I} \subset C(\mathbb{R})$ be a Banach space. If there exists a constant C_3 such that*

$$\left(\sum_{k=-\infty}^{\infty} |f(k)|^2\right)^{\frac{1}{2}} \leq C_3\|f\|_\mathcal{I} \tag{24}$$

for all $f \in \mathcal{I}$, then (23) *converges uniformly on all of $\mathbb{R}$, and consequently in $\mathcal{S}'$, for all $h_T \in \mathcal{PW}_\pi^\infty$ and all $f \in \mathcal{I}$.*

Proof. Assume that (24) is true. Then we have

$$\begin{aligned}&\sum_{k=-\infty}^{\infty} |f(k)h_T(t-k)| \\ &\leq \left(\sum_{k=-\infty}^{\infty} |f(k)|^2\right)^{\frac{1}{2}} \left(\sum_{k=-\infty}^{\infty} |h_T(t-k)|^2\right)^{\frac{1}{2}} \\ &\leq C_3\|f\|_\mathcal{I} \left(\int_{-\infty}^{\infty} |h_T(\tau)|^2 \,\mathrm{d}\tau\right)^{\frac{1}{2}} \\ &= C_3\|f\|_\mathcal{I}\|h_T\|_2 \\ &\leq C_3\|f\|_\mathcal{I}\|h_T\|_{\mathcal{PW}_\pi^\infty}\end{aligned}$$

for all $t \in \mathbb{R}$, where we used Parseval's equality and the fact that $h_T \in \mathcal{PW}_\pi^\infty \subset \mathcal{PW}_\pi^2$ in the third line. Thus, for all $t \in \mathbb{R}$, (23) converges absolutely as N tends to infinity. Let $\epsilon > 0$ be arbitrary but fixed and let

$$F(t) = \sum_{k=-\infty}^{\infty} f(k)h_T(t-k), \quad t \in \mathbb{R}.$$

Because of (24) there exists a natural number $N_1 = N_1(\epsilon)$ such that

$$\left(\sum_{|k|\geq N} |f(k)|^2\right)^{\frac{1}{2}} < \frac{\epsilon}{\|h_T\|_{\mathcal{PW}_\pi^\infty}}$$

for all $N \geq N_1$. Hence, for $N \geq N_1$ and all $t \in \mathbb{R}$ we have

$$\begin{aligned}&\left|F(t) - \sum_{k=-N}^{N} f(k)h_T(t-k)\right| \\ &\leq \sum_{|k|\geq N} |f(k)h_T(t-k)| \\ &\leq \left(\sum_{|k|\geq N} |f(k)|^2\right)^{\frac{1}{2}} \left(\sum_{|k|\geq N} |h_T(t-k)|^2\right)^{\frac{1}{2}} \\ &< \left(\sum_{|k|\geq N} |f(k)|^2\right)^{\frac{1}{2}} \|h_T\|_{\mathcal{PW}_\pi^\infty} \\ &< \epsilon.\end{aligned}$$

It follows that

$$\sup_{t\in\mathbb{R}} \left| F(t) - \sum_{k=-N}^{N} f(k) h_T(t-k) \right| < \epsilon$$

for all $N \geq N_1$. □

For certain subspaces, the condition (24) is also necessary for convergence. In order to state this result, we introduce the concept of amplitude stability on $\mathbb{Z}$. We call a signal space $\mathcal{I} \subset C(\mathbb{R})$ amplitude stable on $\mathbb{Z}$, if there exists a constant C_4 such that for every signal $f \in \mathcal{I}$ and every sequence $\lambda = \{\lambda_k\}_{k\in\mathbb{Z}}$, $\lambda_k \in \mathbb{C}$, $|\lambda_k| = 1$, $k \in \mathbb{Z}$, there exists a signal $f_\lambda \in \mathcal{I}$ such that

$$f_\lambda(k) = \lambda_k f(k), \quad k \in \mathbb{Z},$$

and

$$\|f_\lambda\|_{\mathcal{I}} \leq C_4 \|f\|_{\mathcal{I}}.$$

We have the following theorem, which shows that for subspaces $\mathcal{I} \subset C(\mathbb{R})$ that are amplitude-stable on $\mathbb{Z}$, the condition (24) is not only sufficient for uniform convergence but also necessary.

Theorem 6. *Let $\mathcal{I} \subset C(\mathbb{R})$ be a Banach space that is amplitude stable on $\mathbb{Z}$. Then* (23) *converges uniformly on all of $\mathbb{R}$ for all $h_T \in \mathcal{PW}_\pi^\infty$ and all $f \in \mathcal{I}$ if and only if* (24) *holds for all $f \in \mathcal{I}$.*

For the proof of Theorem 6 we need two lemmas.

Lemma 2. *Let $t \in \mathbb{R}$. If*

$$\lim_{N\to\infty} \sum_{k=-N}^{N} f(k) h_T(t-k) < \infty \tag{25}$$

for all $f \in \mathcal{I}$ and $h_T \in \mathcal{PW}_\pi^\infty$, i.e., if the limit on the left hand side of (25) *exists and is finite for all $f \in \mathcal{I}$ and $h_T \in \mathcal{PW}_\pi^\infty$, then there exists a constant $C_5(t)$ such that*

$$\left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| \leq C_5(t) \|f\|_{\mathcal{I}} \|h_T\|_{\mathcal{PW}_\pi^\infty}$$

for all $f \in \mathcal{I}$, $h_T \in \mathcal{PW}_\pi^\infty$, and $N \in \mathbb{N}$.

Lemma 3. *There exists a constant C_6 such that for every sequence $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2$ there exists a $g \in \mathcal{PW}_\pi^\infty$ such that*

$$\left| \frac{1}{2\pi} \int_{-\pi}^{\pi} \hat{g}(\omega) \, e^{i\omega k} \, d\omega \right| \geq |c_k|, \quad k \in \mathbb{Z},$$

and

$$\|g\|_{\mathcal{PW}_\pi^\infty} \leq C_6 \|c\|_{\ell^2}.$$

The proof of Lemma 2 is given in the appendix, and Lemma 3 can be found in [27], [28].

Proof of Theorem 6. "⇐": This direction has been proved in Theorem 5.

"⇒": Let $\mathcal{I} \subset C(\mathbb{R})$ be a Banach space that is amplitude stable on $\mathbb{Z}$. Let $t \in \mathbb{R}$ be arbitrary but fixed. For $f \in \mathcal{I}$ and $h_T \in \mathcal{PW}_\pi^\infty$, we have according to Lemma 2 that

$$\left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| \leq C_5(t) \|f\|_{\mathcal{I}} \|h_T\|_{\mathcal{PW}_\pi^\infty}$$

for all $N \in \mathbb{N}$. Let $f \in \mathcal{I}$ be arbitrary but fixed. Next, we use the amplitude stability property. Let

$$\lambda_k = e^{-i(\alpha_k(t)+\beta_k)},$$

where $\alpha_k(t) = \arg(h_T(t-k))$ and $\beta_k = \arg(f(k))$. Since $\mathcal{I}$ is amplitude stable on $\mathbb{Z}$, there exists a $f_\lambda \in \mathcal{I}$ with $f_\lambda(k) = \lambda_k f(k)$, $k \in \mathbb{Z}$, and $\|f_\lambda\|_{\mathcal{I}} \leq C_4 \|f\|_{\mathcal{I}}$. Then we have

$$\begin{aligned}\sum_{k=-N}^{N} |f(k)| \cdot |h_T(t-k)| &= \sum_{k=-N}^{N} f_\lambda(k) h_T(t-k) \\ &\leq C_5(t) \|f_\lambda\|_{\mathcal{I}} \|h_T\|_{\mathcal{PW}_\pi^\infty} \\ &\leq C_4 C_5(t) \|f\|_{\mathcal{I}} \|h_T\|_{\mathcal{PW}_\pi^\infty}\end{aligned}$$

for all $h_T \in \mathcal{PW}_\pi^\infty$. According to Lemma 3 there exists a constant C_6 such that for every sequence $c = \{c_k\}_{k\in\mathbb{Z}} \in \ell^2$ there exists a $h_c \in \mathcal{PW}_\pi^\infty$ such that

$$|h_c(t-k)| \geq |c_k|, \quad k \in \mathbb{Z},$$

and

$$\|h_c\|_{\mathcal{PW}_\pi^\infty} \leq C_6 \|c\|_{\ell^2}.$$

It follows that

$$\begin{aligned}\sum_{k=-N}^{N} |f(k)| \cdot |c_k| &\leq \sum_{k=-N}^{N} |f(k)| \cdot |h_c(t-k)| \\ &\leq C_4 C_5(t) C_6 \|f\|_{\mathcal{I}} \|c\|_{\ell^2}.\end{aligned}$$

For $N \in \mathbb{N}$ we choose

$$|c_k^{(N)}| = |f(k)| \left(\sum_{k=-N}^{N} |f(k)|^2 \right)^{-\frac{1}{2}}$$

for $|k| \leq N$ and $c_k^{(N)} = 0$ for $|k| > N$. Since

$$\begin{aligned}\sum_{k=-N}^{N} |f(k)| \cdot |c_k^{(N)}| &= \frac{\sum_{k=-N}^{N} |f(k)|^2}{\left(\sum_{k=-N}^{N} |f(k)|^2 \right)^{-\frac{1}{2}}} \\ &= \left(\sum_{k=-N}^{N} |f(k)|^2 \right)^{\frac{1}{2}},\end{aligned}$$

we obtain

$$\left(\sum_{k=-N}^{N} |f(k)|^2 \right)^{\frac{1}{2}} \leq C_4 C_5(t) C_6 \|f\|_{\mathcal{I}}. \tag{26}$$

The right hand side of (26) does not depend on N. Therefore, it follows that

$$\left(\sum_{k=-\infty}^{\infty} |f(k)|^2 \right)^{\frac{1}{2}} \leq C_4 C_5(t) C_6 \|f\|_{\mathcal{I}},$$

which completes the proof. □

In the rest of this section we show that in the scale of Bernstein spaces, $\mathcal{B}_\pi^p$, $1 \leq p \leq \infty$, $p = 2$ is the largest p for which we have pointwise convergence for all $h_T \in \mathcal{PW}_\pi^\infty$ and all signals from the signal space. In other words, $\mathcal{B}_\pi^2$ is the largest space for which (23) converges pointwise for all

$h_T \in \mathcal{PW}_\pi^\infty$ and all $f \in \mathcal{B}_\pi^2$ as N tends to infinity. For all $p > 2$ we have divergence.

The convergence for $\mathcal{B}_\pi^p$, $1 \le p \le 2$, follows from the fact that $\mathcal{B}_\pi^p \subset \mathcal{B}_\pi^2$ for all $1 \le p \le 2$, and from the convergence of (23) for $f \in \mathcal{B}_\pi^2$, which is a consequence of the Cauchy–Schwarz inequality.

In order to prove the divergence claim, we will construct, for arbitrary $t \in \mathbb{R}$, an $h_{T_c} \in \mathcal{PW}_\pi^\infty$ and a signal f_λ that is in $\mathcal{B}_\pi^p$ for all $p > 2$, such that

$$\lim_{N\to\infty} \sum_{k=-N}^{N} f_\lambda(k) h_{T_c}(t-k) = \infty.$$

This shows that we have convergence for all signals if $1 \le p \le 2$ and divergence for certain signals if $p > 2$.

Let $t \in \mathbb{R}$ be arbitrary but fixed. Further, let $\{c_k\}_{k\in\mathbb{Z}} \in \ell^2$ be some sequence with $\{c_k\}_{k\in\mathbb{Z}} \notin \ell^q$ for all $q < 2$; for example, choose

$$c_k = \frac{1}{(|k|+1)^{\frac{1}{2}} \log(2+|k|)}, \quad k \in \mathbb{Z}.$$

Then, according to Lemma 3, there exists an $h_{T_c} \in \mathcal{PW}_\pi^\infty$ such that

$$|h_{T_c}(t-k)| \ge c_k, \quad k \in \mathbb{Z}.$$

Let $\alpha = \{\alpha_k\}_{k\in\mathbb{Z}}$ be defined by

$$\alpha_k = \frac{1}{(1+|k|)^{\frac{1}{2}}}, \quad k \in \mathbb{Z}.$$

Then we have $\alpha \in \ell^p$ for all $p > 2$. Further, a straight forward calculation shows that

$$\sum_{k=-N}^{N} \alpha_k c_k > 2 \log\left(\frac{\log(N+2)}{\log(2)}\right).$$

Let

$$\lambda_k = \mathrm{e}^{-i\gamma_k(t)},$$

where $\gamma_k(t) = \arg(h_{T_c}(t-k))$, and let

$$f_\lambda(t) = \sum_{k=-\infty}^{\infty} \alpha_k \lambda_k \frac{\sin(\pi(t-k))}{\pi(t-k)}, \quad t \in \mathbb{R}.$$

Then we have $f_\lambda \in \mathcal{B}_\pi^p$ for all $p > 2$, according to the Plancherel–Pólya theorem [29, p. 152]. Further, it follows that

$$\begin{aligned}\sum_{k=-N}^{N} f_\lambda(k) h_{T_c}(t-k) &= \sum_{k=-N}^{N} \alpha_k \lambda_k h_{T_c}(t-k) \\ &\ge \sum_{k=-N}^{N} \alpha_k c_k \\ &> 2\log\left(\frac{\log(N+2)}{\log(2)}\right),\end{aligned}$$

and consequently that

$$\lim_{N\to\infty} \sum_{k=-N}^{N} f_\lambda(k) h_{T_c}(t-k) = \infty.$$

Thus, we have strong divergence for $t \in \mathbb{R}$.

The above calculation implies that we also have divergence for all mentioned windowing functions, such as triangular window, trapezoidal window, and cosine roll-off window.

X. Proofs

In this section we prove the central Lemma 1, and several necessary auxiliary lemmas. Let

$$S_N f = \sum_{k=-N}^{N} c_k \, \mathrm{e}^{i\omega k}, \quad \omega \in [-\pi, \pi],$$

denote the N-th partial sum of the Fourier series of a function $f \in C([-\pi,\pi])$, where

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\omega) \, \mathrm{e}^{-i\omega k} \, \mathrm{d}\omega, \quad k \in \mathbb{Z},$$

are the usual Fourier coefficients.

Lemma 4. *Let $\epsilon \in (0,\pi)$ and $K > 0$. There exists a function $q_{\epsilon,K}$ such that*

1) *$q_{\epsilon,K} \in C_0^\infty[-\epsilon,\epsilon]$,*
2) *there exists a natural number $N_0 = N_0(\epsilon, K)$ such that $|(S_{N_0} q_{\epsilon,K})(0)| > K$, and*
3) *$|(S_N q_{\epsilon,K})(\omega)| \le 1$ for all $2\epsilon \le |\omega| \le \pi$ and all $N \in \mathbb{N}$.*

Proof. Let $\epsilon \in (0,\pi)$ be arbitrary but fixed. The partial sum of the Fourier series of a function $g \in C_0^\infty$ is given by

$$(S_N g)(\omega) = \frac{1}{\pi} \int_{-\pi}^{\pi} g(\omega_1) \frac{\sin\left(\frac{2N+1}{2}(\omega-\omega_1)\right)}{2\sin\left(\frac{1}{2}(\omega-\omega_1)\right)} \, \mathrm{d}\omega_1. \quad (27)$$

For $N \in \mathbb{N}$ let

$$u_{N,\epsilon}(\omega) = \begin{cases} \operatorname{sgn}\left(\frac{\sin\left(\frac{2N+1}{2}\omega\right)}{2\sin\left(\frac{1}{2}\omega\right)}\right), & |\omega| \le \epsilon, \\ 0, & \epsilon < |\omega| \le \pi. \end{cases}$$

Then we have

$$(S_N u_{N,\epsilon})(0) = \frac{1}{\pi} \int_{-\epsilon}^{\epsilon} \left| \frac{\sin\left(\frac{2N+1}{2}\omega_1\right)}{2\sin\left(\frac{1}{2}\omega_1\right)} \right| \mathrm{d}\omega_1.$$

According to the behavior of the L^1-norm of the Dirichlet kernel [30, p. 42], it follows that

$$\lim_{N\to\infty} \frac{1}{\pi} \int_{-\epsilon}^{\epsilon} \left| \frac{\sin\left(\frac{2N+1}{2}\omega_1\right)}{2\sin\left(\frac{1}{2}\omega_1\right)} \right| \mathrm{d}\omega_1 = \infty.$$

Hence, for every $K > 0$ there exists a natural number $N_0 = N_0(\epsilon, K)$ such that

$$(S_{N_0} u_{N_0,\epsilon})(0) > K.$$

Clearly, the functions $u_{N,\epsilon}$ are not continuous. However, using the same technique as in Section 1.3., Chapter II of [31, p. 69], each function $u_{N,\epsilon}$ can be approximated by a function $\tilde{u}_{N,\epsilon} \in C_0^\infty[-\epsilon,\epsilon]$ with $\|\tilde{u}_{N,\epsilon}\|_\infty \le 1$, such that we have

$$(S_{N_0} \tilde{u}_{N_0,\epsilon})(0) > K.$$

We set $q_{\epsilon,K} = \tilde{u}_{N_0,\epsilon}$. Further, for $2\epsilon < |\omega| \le \pi$ and all $N \in \mathbb{N}$, we have

$$\begin{aligned}|(S_N q_{\epsilon,K})(\omega)| &\le \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \left| \frac{\sin\left(\frac{2N+1}{2}(\omega-\omega_1)\right)}{\sin\left(\frac{1}{2}(\omega-\omega_1)\right)} \right| \mathrm{d}\omega_1 \\ &\le \frac{1}{2\pi} \int_{-\epsilon}^{\epsilon} \frac{1}{|\sin\left(\frac{1}{2}(\omega-\omega_1)\right)|} \, \mathrm{d}\omega_1 \\ &\le \frac{\epsilon}{\pi \sin(\frac{\epsilon}{2})} \\ &\le 1,\end{aligned}$$

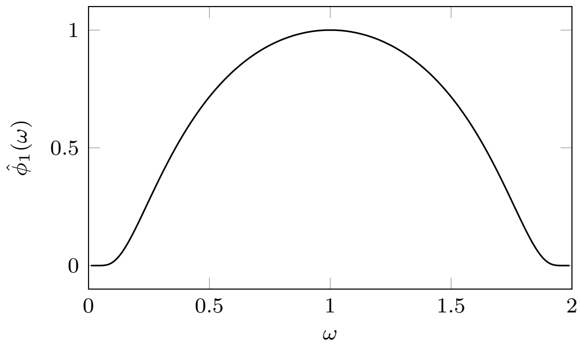

Fig. 1. Plot of the function $\hat{\phi}_1$.

because $\sin(x) \geq 2x/\pi$ for $0 \leq x \leq \pi/2$. □

For the next lemma we need to define a function $\hat{\phi}$ with certain properties. Let $\hat{\phi}$ be a function in $C_0^\infty[0,2]$ that satisfies $0 < \hat{\phi}(\omega) \leq 1$ for all $\omega \in (0,2)$. A function $\hat{\phi}$ that satisfies these properties is given by

$$\hat{\phi}_1(\omega) = \begin{cases} \mathrm{e}^{1-\frac{1}{1-(\omega-1)^2}}, & 0 < \omega < 2, \\ 0, & \text{otherwise.} \end{cases}$$

$\hat{\phi}_1$ is illustrated in Fig. 1. Note that $\phi = \mathcal{F}^{-1}\hat{\phi} \in \mathcal{S}$ for any $\hat{\phi}$ with above properties.

Lemma 5. *There exists a continuous function $g \in C[-\pi,\pi]$, such that*

1) *$g(\omega) = 0$ for all $\omega \in [-\pi,\pi] \setminus [0,3]$,*
2) *for every $\delta > 0$, g is infinitely often differentiable on $[\delta,\pi]$, and*
3) *there exists a strictly monotonically decreasing sequence of real numbers $\{w_n\}_{n\in\mathbb{N}}$ with $\lim_{n\to\infty} w_n = 0$, such that for*

$$\hat{\phi}_n(\omega) = \hat{\phi}(\omega - \omega_n)$$

we have

$$\lim_{n\to\infty} \left(\sup_{N\in\mathbb{N}} |(S_N g)(2\omega_n)\hat{\phi}_n(2\omega_n)|\right) = \infty.$$

Proof. We construct the function g inductively and use Lemma 4. Let $\omega_1 = 1$, $\epsilon_1 = 1/4$, and $K_1 = 2$. We set

$$\hat{\phi}_1(\omega) = \hat{\phi}(\omega - \omega_1)$$

and

$$g_1(\omega) = q_{\epsilon_1,K_1}(\omega - 2\omega_1).$$

Further, let $c_1 = \hat{\phi}_1(2\omega_1) = \hat{\phi}(\omega_1)$. We have $0 < c_1 \leq 1$.

We set $\omega_2 = \omega_1/4$. Further, let

$$\hat{\phi}_2(\omega) = \hat{\phi}(\omega - \omega_2)$$

and $c_2 = \hat{\phi}_2(2\omega_2) = \hat{\phi}(\omega_2)$. We have $0 < c_2 \leq 1$. We choose a K_2 with $K_2 > 2^2 2^2/c_2$, and $\epsilon_2 = \omega_2/4$. Let

$$g_2(\epsilon) = \frac{1}{4} q_{\epsilon_2,K_2}(\omega - 2\omega_2) + g_1(\omega).$$

Assume that for $1 \leq l \leq n$ we have defined the functions $\hat{\phi}_l$ and g_l as well as the numbers ω_l and c_l. Then we set $\omega_{n+1} = \omega_n/4$. Further, let

$$\hat{\phi}_{n+1}(\omega) = \hat{\phi}(\omega - \omega_{n+1})$$

and $c_{n+1} = \hat{\phi}_{n+1}(2\omega_{n+1}) = \hat{\phi}(\omega_{n+1})$. We have $0 < c_{n+1} \leq 1$. We choose a K_{n+1} with $K_{n+1} > 2^{n+1}(n+1)^2/c_{n+1}$, and $\epsilon_{n+1} = \omega_{n+1}/4$. We set

$$g_{n+1}(\omega) = \frac{1}{(n+1)^2} q_{\epsilon_{n+1},K_{n+1}}(\omega - 2\omega_{n+1}) + g_n(\omega).$$

Following this procedure, we have constructed the function

$$g(\omega) = \sum_{n=1}^{\infty} \frac{1}{n^2} q_{\epsilon_n,K_n}(\omega - 2\omega_n). \tag{28}$$

Since

$$\left| \frac{1}{n^2} q_{\epsilon_n,K_n}(\omega - 2\omega_n) \right| \leq \frac{1}{n^2},$$

we see that the series in (28) converges absolutely and uniformly. Due to the uniform convergence, the function g is continuous. Further, g is concentrated on the interval $[0,3]$. Let $\delta > 0$ be arbitrary. For $\omega \in [\delta,\pi]$, only finitely many summands in the series (28) are different from zero. Since each summand is a C_0^∞ function, it follows that g is infinitely often differentiable on $[\delta,\pi]$.

Let $n \in \mathbb{N}$ be arbitrary. We prove the third assertion of the lemma next. For $N \in \mathbb{N}$ we have

$$\begin{aligned}
&(S_N g)(2\omega_n)\hat{\phi}_n(2\omega_n) \\
&= c_n (S_N g)(2\omega_n) \\
&= \frac{c_n}{n^2}(S_N q_{\epsilon_n,K_n}(\,\cdot\, - 2\omega_n))(2\omega_n) \\
&\quad + c_n \sum_{m=1}^{n-1} \frac{1}{m^2}(S_N q_{\epsilon_m,K_m}(\,\cdot\, - 2\omega_m))(2\omega_n) \\
&\quad + c_n \sum_{m=n+1}^{\infty} \frac{1}{m^2}(S_N q_{\epsilon_m,K_m}(\,\cdot\, - 2\omega_m))(2\omega_n) \\
&= \frac{c_n}{n^2}(S_N q_{\epsilon_n,K_n})(0) \\
&\quad + c_n \sum_{m=1}^{n-1} \frac{1}{m^2}(S_N q_{\epsilon_m,K_m})(2(\omega_n - \omega_m)) \\
&\quad + c_n \sum_{m=n+1}^{\infty} \frac{1}{m^2}(S_N q_{\epsilon_m,K_m})(2(\omega_n - \omega_m)).
\end{aligned}$$

For $m < n$ we have

$$\epsilon_m = \frac{\omega_m}{4} = \frac{1}{4^m} < \frac{3}{4^m} \leq \omega_m - \omega_n$$

and consequently $2\epsilon_m < |2(\omega_n - \omega_m)| < \pi$. For $m > n$ we have

$$\epsilon_m = \frac{\omega_m}{4} = \frac{1}{4^m} < \frac{3}{4^n} \leq \omega_n - \omega_m$$

and consequently $2\epsilon_m < |2(\omega_n - \omega_m)| < \pi$. Hence, it follows from item 3 of Lemma 4 that

$$\left| \sum_{m=1}^{n-1} \frac{1}{m^2}(S_N q_{\epsilon_m,K_m})(2(\omega_n - \omega_m)) \right| \leq \sum_{m=1}^{n-1} \frac{1}{m^2} \leq \frac{\pi^2}{6}$$

and

$$\left| \sum_{m=n+1}^{\infty} \frac{1}{m^2}(S_N q_{\epsilon_m,K_m})(2(\omega_n - \omega_m)) \right| \leq \sum_{m=n+1}^{\infty} \frac{1}{m^2} \leq \frac{\pi^2}{6}.$$

Thus, for $N \in \mathbb{N}$, we obtain

$$\left|(S_N g)(2\omega_n)\hat{\phi}_n(2\omega_n) - \frac{c_n}{n^2}(S_N q_{\epsilon_n,K_n})(0)\right| \le c_n \frac{\pi^2}{3}.$$

According to item 2 in Lemma 4 there exists an $N_0 = N_0(\epsilon_n, K_n)$ such that

$$|(S_{N_0} q_{\epsilon_n,K_n})(0)| > K_n > \frac{2^n n^2}{c_n}.$$

Hence, we obtain

$$\begin{aligned}|(S_{N_0} g)(2\omega_n)\hat{\phi}_n(2\omega_n)| &\ge \left|\frac{c_n}{n^2}(S_{N_0} q_{\epsilon_n,K_n})(0)\right| - c_n\frac{\pi^2}{3} \\ &> 2^n - c_n\frac{\pi^2}{3},\end{aligned}$$

which completes the proof, because $n \in \mathbb{N}$ was arbitrary. □

Lemma 6. *For all $g \in C[-\pi,\pi]$, all $\hat{\phi} \in C_0^\infty[0,3]$, and all $N \in \mathbb{N}$, we have*

$$\|S_N(g\hat{\phi}) - \hat{\phi}S_N g\|_C \le \pi\|g\|_C\|\hat{\phi}'\|_C.$$

Proof. Let $g \in C[-\pi,\pi]$ and $\hat{\phi} \in C_0^\infty[0,3]$ be arbitrary. For convenience, we introduce the 2π-periodic extension of $\hat{\phi}$, which we denote by $\hat{\phi}_p$. Let $\omega \in [-\pi,\pi]$ be arbitrary but fixed. Using (27), we obtain

$$\begin{aligned}&|(S_N(g\hat{\phi}))(\omega) - \hat{\phi}(\omega)(S_N g)(\omega)| \\ &= \left|\frac{1}{\pi}\int_{-\pi}^{\pi}\left(g(\omega_1)\hat{\phi}(\omega_1)\frac{\sin\left(\frac{2N+1}{2}(\omega-\omega_1)\right)}{2\sin\left(\frac{1}{2}(\omega-\omega_1)\right)}\right.\right. \\ &\qquad \left.\left. - g(\omega_1)\hat{\phi}(\omega)\frac{\sin\left(\frac{2N+1}{2}(\omega-\omega_1)\right)}{2\sin\left(\frac{1}{2}(\omega-\omega_1)\right)}\right) \mathrm{d}\omega_1\right| \\ &\le \frac{1}{\pi}\int_{-\pi}^{\pi}|g(\omega_1)|\left|\frac{\hat{\phi}(\omega_1)-\hat{\phi}(\omega)}{2\sin\left(\frac{1}{2}(\omega-\omega_1)\right)}\right| \\ &\qquad \left|\sin\left(\frac{2N+1}{2}(\omega-\omega_1)\right)\right| \mathrm{d}\omega_1 \\ &\le \|g\|_C\frac{1}{\pi}\int_{-\pi}^{\pi}\left|\frac{\hat{\phi}(\omega_1)-\hat{\phi}(\omega)}{2\sin\left(\frac{1}{2}(\omega-\omega_1)\right)}\right| \mathrm{d}\omega_1. \end{aligned} \tag{29}$$

Since

$$|\hat{\phi}_p(\omega_1) - \hat{\phi}_p(\omega)| \le \|\hat{\phi}'\|_C|\omega-\omega_1|$$

according to the mean value theorem, we see that

$$\begin{aligned}&\int_{-\pi}^{\pi}\left|\frac{\hat{\phi}(\omega_1)-\hat{\phi}(\omega)}{2\sin\left(\frac{1}{2}(\omega-\omega_1)\right)}\right| \mathrm{d}\omega_1 \\ &= \int_{-\pi}^{\pi}\left|\frac{\hat{\phi}_p(\omega_1)-\hat{\phi}_p(\omega)}{2\sin\left(\frac{1}{2}(\omega-\omega_1)\right)}\right| \mathrm{d}\omega_1 \\ &= \int_{\omega-\pi}^{\omega+\pi}\left|\frac{\hat{\phi}_p(\omega_1)-\hat{\phi}_p(\omega)}{2\sin\left(\frac{1}{2}(\omega-\omega_1)\right)}\right| \mathrm{d}\omega_1 \\ &\le \|\hat{\phi}'\|_C\frac{1}{2}\int_{\omega-\pi}^{\omega+\pi}\left|\frac{\omega-\omega_1}{\sin\left(\frac{1}{2}(\omega-\omega_1)\right)}\right| \mathrm{d}\omega_1 \\ &= \|\hat{\phi}'\|_C\frac{1}{2}\int_{-\pi}^{\pi}\left|\frac{\tau}{\sin\left(\frac{\tau}{2}\right)}\right| \mathrm{d}\tau. \end{aligned} \tag{30}$$

Combining (29) and (30), we obtain

$$\begin{aligned}&|(S_N(g\hat{\phi}))(\omega) - \hat{\phi}(\omega)(S_N g)(\omega)| \\ &\le \|g\|_C\|\hat{\phi}'\|_C\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|\frac{\tau}{\sin\left(\frac{\tau}{2}\right)}\right| \mathrm{d}\tau \\ &\le \pi\|g\|_C\|\hat{\phi}'\|_C,\end{aligned}$$

because $|\sin(\tau/2)| \ge |\tau/\pi|$ for $-\pi \le \tau \le \pi$. Since $\omega \in [-\pi,\pi]$ was arbitrary, the assertion of the theorem follows after taking the supremum on both sides of the inequality. □

Lemma 7. *Let g, $\{\hat{\phi}_n\}_{n\in\mathbb{N}}$, and $\{\omega_n\}_{n\in\mathbb{N}}$ be defined as in the proof of Lemma 5. Then we have*

$$\lim_{n\to\infty}\left(\sup_{N\in\mathbb{N}}(S_N(g\hat{\phi}_n))(2\omega_n)\right) = \infty.$$

Proof. Since we have

$$\hat{\phi}_n(\omega) = \hat{\phi}(\omega-\omega_n),$$

it follows that $\|\phi_n'\|_C = \|\phi'\|_C$. Using Lemma 6, we see that

$$\begin{aligned}\|S_N(g\hat{\phi}_n) - \hat{\phi}_n S_N g\|_C &\le \pi\|g\|_C\|\hat{\phi}_n'\|_C \\ &\le \pi\|g\|_C\|\hat{\phi}'\|_C\end{aligned} \tag{31}$$

for all $n \in \mathbb{N}$ and $N \in \mathbb{N}$. Since the right hand side of (31) is independent of n and N, the assertion follows directly from Lemma 5. □

Lemma 8. *Let $\{\hat{\phi}_n\}_{n\in\mathbb{N}}$ be defined as in the proof of Lemma 5. There exists a function $\hat{h}_T \in C[-\pi,\pi]$ that is concentrated on $[0,3]$ such that*

1) *For every $\delta > 0$, $\hat{h}_T$ is infinitely often differentiable on $[\delta,\pi]$,*
2) *for all $n \in \mathbb{N}$ we have*

$$\sum_{k=-\infty}^{\infty}|c_k(h_T,\phi_n)| < \infty,$$

and

3)

$$\lim_{n\to\infty}\sum_{k=-\infty}^{\infty}|c_k(h_T,\phi_n)| = \infty.$$

Proof. We choose $\hat{h}_T = g$, where g is the function that was constructed in the proof of Lemma 5. Then $\hat{h}_T$ is concentrated on $[0,3]$ and satisfies item 1) of the assertion.

Let $n \in \mathbb{N}$ be arbitrary. The function $\hat{\phi}_n$ is concentrated on $[\omega_n, \omega_n+2]$. Hence, for the function

$$v_n := \hat{h}_T\hat{\phi}_n$$

we have $v_n \in C_0^\infty[\omega_n,\omega_n+2]$. It follows that the Fourier coefficients

$$c_k(h_T,\phi_n) = \frac{1}{2\pi}\int_{-\pi}^{\pi}v_n(\omega)\,\mathrm{e}^{i\omega k}\,\mathrm{d}\omega$$

satisfy

$$\lim_{k\to\infty}|k|^r|c_k(h_T,\phi_n)| = 0$$

for all $r \in \mathbb{N}$ [32, p. 196, Theorem 3.3.9]. Thus, we have

$$\sum_{k=-\infty}^{\infty}|c_k(h_T,\phi_n)| < \infty,$$

which proves item 2) of the assertion.

Item 3) remains to be proved. The N-th partial sum of the Fourier series of the function v_n is given by

$$S_N(\hat{h}_T\hat{\phi}_n)(\omega) = \sum_{k=-N}^{N} c_k(h_T,\phi_n)\,\mathrm{e}^{ik\omega}$$

and we have

$$|S_N(\hat{h}_T\hat{\phi}_n)(\omega)| = \left|\sum_{k=-N}^{N} c_k(h_T,\phi_n)\,\mathrm{e}^{ik\omega}\right| \leq \sum_{k=-\infty}^{\infty} |c_k(h_T,\phi_n)|. \quad (32)$$

Since (32) is valid for all $N\in\mathbb{N}$ and all $\omega\in[-\pi,\pi]$, it follows that

$$\sup_{N\in\mathbb{N}} |S_N(\hat{h}_T\hat{\phi}_n)(2\omega_n)| \leq \sum_{k=-\infty}^{\infty} |c_k(h_T,\phi_n)|,$$

and the assertion follows directly from Lemma 7. □

Lemma 9. *Let $\{\hat{\phi}_n\}_{n\in\mathbb{N}}$ be defined as in the proof of Lemma 5. Then $\{\phi_n\}_{n\in\mathbb{N}}$ converges to ϕ in $\mathcal{S}$.*

Proof. We have $\hat{\phi}_n(\omega) = \hat{\phi}(\omega-\omega_n)$ and $\lim_{n\to\infty}\omega_n = 0$. Hence, it follows that $\phi_n(t) = \phi(t)\,\mathrm{e}^{i\omega_n t}$ and

$$|\phi(t)-\phi_n(t)| = |\phi(t)||1-\mathrm{e}^{i\omega_n t}|.$$

We will show that for arbitrary $a\in\mathbb{N}_0$ we have

$$\lim_{n\to\infty}\sup_{t\in\mathbb{R}} |t|^a|\phi(t)-\phi_n(t)| = 0.$$

Let $\epsilon>0$ be arbitrary. Then there exist a $T_0 = T_0(\epsilon,a)$ such that

$$|\phi(t)||t|^a < \frac{\epsilon}{2}$$

for all $|t|\geq T_0$. Thus, for $|t|\geq T_0$ and all $n\in\mathbb{N}$, we have

$$\begin{aligned} |t|^a|\phi(t)-\phi_n(t)| &= |t|^a|\phi(t)||1-\mathrm{e}^{i\omega_n t}| \\ &\leq 2|t|^a|\phi(t)| \\ &< \epsilon. \end{aligned}$$

For $|t|\leq T_0$ we have

$$|1-\mathrm{e}^{i\omega_n t}| \leq \omega_n|t| \leq \omega_n T_0.$$

Hence, there exists a $n_0 = n_0(\epsilon)$ such that

$$|1-\mathrm{e}^{i\omega_n t}| < \frac{\epsilon}{C_a}$$

for all $n\geq n_0$ and all $|t|\leq T_0$, where

$$C_a = \sup_{t\in\mathbb{R}} |t|^a|\phi(t)|.$$

It follows that

$$\sup_{t\in\mathbb{R}} |t|^a|\phi(t)-\phi_n(t)| < \epsilon$$

for all $n\geq n_0$. The same proof works for all derivatives $\phi^{(b)}$ and the corresponding sequences $\{\phi_n^{(b)}\}_{n\in\mathbb{N}}$, $b\geq 1$. □

Lemma 10. *For $h_T\in\mathcal{PW}_\pi^\infty$ and $\phi\in\mathcal{S}$ we have*

$$\sup_{\substack{f\in\mathcal{C}\\ \|f\|_{\mathcal{C}}\leq 1}} \left|\sum_{k=-\infty}^{\infty} f(k)c_k(h_T,\phi)\right| = \sum_{k=-\infty}^{\infty} |c_k(h_T,\phi)|.$$

Proof. Let $h_T\in\mathcal{PW}_\pi^\infty$ and $\phi\in\mathcal{S}$ be arbitrary but fixed. Since, for all $f\in\mathcal{C}$, we have

$$\left|\sum_{k=-\infty}^{\infty} f(k)c_k(h_T,\phi)\right| \leq \|f\|_{\mathcal{C}} \sum_{k=-\infty}^{\infty} |c_k(h_T,\phi)|,$$

it follows that

$$\sup_{\substack{f\in\mathcal{C}\\ \|f\|_{\mathcal{C}}\leq 1}} \left|\sum_{k=-\infty}^{\infty} f(k)c_k(h_T,\phi)\right| \leq \sum_{k=-\infty}^{\infty} |c_k(h_T,\phi)|. \quad (33)$$

For $N\in\mathbb{N}$, let

$$a_k^{(N)} = \begin{cases} \operatorname{sgn}(|c_k(h_T,\phi)|), & |k|\leq N, \\ 0, & |k|>N. \end{cases}$$

Next, we will show that there exists a function $g_N\in\mathcal{C}$ such that $g_N(k) = a_k^{(N)}$, $k\in\mathbb{Z}$, and $\|g_N\|_{\mathcal{C}}\leq 1$. This part of the proof follows closely the proof of Lemma 1 in [6], however, for completeness we include it here. For $k\in\mathbb{Z}$, let $l_k = 2^{-|k|}/3$ and define

$$d_k(t) = \begin{cases} 0, & |t-k|\geq l_k, \\ 1-l_k^{-1}|t-k|, & |t-k|<l_k. \end{cases}$$

Let

$$g_N(t) = \sum_{k=-N}^{N} a_k^{(N)} d_k(t), \quad t\in\mathbb{R}. \quad (34)$$

Clearly, we have $g_N(k) = a_k^{(N)}$ for all $k\in\mathbb{Z}$. Further, we have

$$\int_{-\infty}^{\infty} |g_N(t)|\,\mathrm{d}t \leq \|a^{(N)}\|_{c_0}\int_{-\infty}^{\infty}\sum_{k=-N}^{N}|d_k(t)|\,\mathrm{d}t \leq 1,$$

because

$$\begin{aligned} \int_{-\infty}^{\infty}\sum_{k=-N}^{N}|d_k(t)|\,\mathrm{d}t &= \sum_{k=-N}^{N}\int_{-\infty}^{\infty} d_k(t)\,\mathrm{d}t \\ &= \sum_{k=-N}^{N} l_k \\ &\leq \frac{1}{3}\sum_{k=-\infty}^{\infty}\frac{1}{2^{|k|}} \\ &= 1. \end{aligned}$$

Moreover, we have

$$|g_N(t)| \leq \sum_{k=-N}^{N} |a_k^{(N)}||d_k(t)| \leq \|a^{(N)}\|_{c_0} = 1$$

and

$$\lim_{|t|\to\infty} |g_N(t)| = 0.$$

Hence, we see that $g_N \in \mathcal{C}$ with $\|g_N\|_{\mathcal{C}} \leq 1$ and $g_N(k) = a_k^{(N)}$, $k \in \mathbb{Z}$. It follows that

$$\sup_{\substack{f\in\mathcal{C}\\ \|f\|_{\mathcal{C}}\leq 1}} \left|\sum_{k=-\infty}^{\infty} f(k)c_k(h_T,\phi)\right| \geq \left|\sum_{k=-\infty}^{\infty} g_N(k)c_k(h_T,\phi)\right| = \sum_{k=-N}^{N} |c_k(h_T,\phi)|.$$

Since $N \in \mathbb{N}$ was arbitrary, we can take the limit $N \to \infty$ to obtain

$$\sup_{\substack{f\in\mathcal{C}\\ \|f\|_{\mathcal{C}}\leq 1}} \left|\sum_{k=-\infty}^{\infty} f(k)c_k(h_T,\phi)\right| \geq \sum_{k=-\infty}^{\infty} |c_k(h_T,\phi)|. \quad (35)$$

Combining (33) and (35) completes the proof. □

Now we are in the position to prove Lemma 1.

Proof of Lemma 1. According to Lemma 8 there exists a sequence $\{\phi_n^*\}_{n\in\mathbb{N}} \subset \mathcal{S}$ and a function $h_T^* \in \mathcal{PW}_\pi^\infty$ such that

$$\sum_{k=-\infty}^{\infty} |c_k(h_T^*,\phi_n^*)| < \infty \quad (36)$$

for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \sum_{k=-\infty}^{\infty} |c_k(h_T^*,\phi_n^*)| = \infty. \quad (37)$$

Item 1) of the assertion follows directly from (36). Further, item 2) follows from Lemma 9. Item 3), i.e., the fact that there exists a signal $f^* \in \mathcal{C}$ such that

$$\limsup_{n\to\infty} \left|\sum_{k=-\infty}^{\infty} f^*(k)c_k(h_T^*,\phi_n^*)\right| = \infty,$$

remains to be proved. To this end, we consider the bounded linear functionals $\Gamma_n : \mathcal{C} \to \mathbb{C}$, $n \in \mathbb{N}$, defined by

$$\Gamma_n f = \sum_{k=-\infty}^{\infty} f(k)c_k(h_T^*,\phi_n^*).$$

According to Lemma 10 we have

$$\|\Gamma_n\| = \sup_{\substack{f\in\mathcal{C}\\ \|f\|_{\mathcal{C}}\leq 1}} \left|\sum_{k=-\infty}^{\infty} f(k)c_k(h_T^*,\phi_n^*)\right| = \sum_{k=-\infty}^{\infty} |c_k(h_T^*,\phi_n^*)|$$

for all $n \in \mathbb{N}$. Hence, from (37) we see that $\sup_{n\in\mathbb{N}}\|\Gamma_n\|_\infty = \infty$. The Banach–Steinhaus theorem [33, p. 98] implies that there exists a signal $f^* \in \mathcal{C}$ such that

$$\limsup_{n\to\infty} |\Gamma_n f^*| = \limsup_{n\to\infty} \left|\sum_{k=-\infty}^{\infty} f^*(k)c_k(h_T^*,\phi_n^*)\right| = \infty,$$

which completes the proof. □

Proof of Theorem 2. Let $h_T^* \in \mathcal{PW}_\pi^\infty$ and $\{\phi_n^*\}_{n\in\mathbb{N}}$ be as in the definition of Lemma 1. Then $f \mapsto G(f,h_T^*,\phi_n^*)$, $n \in \mathbb{N}$, are nothing else than the bounded linear functionals Γ_n that we used in the proof of Lemma 1. Using the same reasoning as there, and the Banach–Steinhaus theorem [33, p. 98] in a slightly stronger version, we see that the set of signals $f \in \mathcal{C}$ such that

$$\limsup_{n\to\infty} |G(f,h_T^*,\phi_n^*)| = \infty,$$

is a residual set. □

In the proof of Theorem 3 we need several functions from the space $\mathcal{PW}_\pi^\infty$. Since each $h_T \in \mathcal{PW}_\pi^\infty$ defines a stable LTI system T, we denote these functions by h_{T_1} and h_{T_2}.

Proof of Theorem 3. In the proof of Lemma 1 we constructed functions $h_T^* \in \mathcal{PW}_\pi^\infty$ and $\phi^* \in \mathcal{S}$ such that

$$\lim_{N\to\infty} \sum_{k=-N}^{N} |c_k(h_T^*,\phi^*)| = \infty. \quad (38)$$

For all $k \in \mathbb{Z}$, the mappings

$$c_k(\,\cdot\,,\phi^*) \colon \mathcal{PW}_\pi^\infty \to \mathbb{C}$$

are continuous linear functionals. Hence, for all $N \in \mathbb{N}$, $\Theta_N \colon \mathcal{PW}_\pi^\infty \to \mathbb{R}$, defined by

$$\Theta_N(h_T) = \sum_{k=-N}^{N} |c_k(h_T,\phi^*)|$$

is a continuous real-valued functional. Suppose that there exists a set $\mathcal{K}$ of the second category in $\mathcal{PW}_\pi^\infty$ with

$$\lim_{N\to\infty} \Theta_N(h_T) < \infty \quad (39)$$

for all $h_T \in \mathcal{K}$. The limit exists, because $\Theta_{N+1}(h_T) \geq \Theta_N(h_T)$, $N \in \mathbb{N}$. According to the generalized uniform boundedness theorem (see Theorem 7 in the appendix) there exist a $\delta > 0$, an $h_{T_1} \in \mathcal{PW}_\pi^\infty$, and a constant C_7, such that

$$\Theta_N(h_T) \leq C_7 \quad (40)$$

for all $h_T \in \mathcal{PW}_\pi^\infty$ with $\|h_T - h_{T_1}\|_{\mathcal{PW}_\pi^\infty} < \delta$. Let

$$h_{T_2} = h_{T_1} + \frac{\delta}{2}\frac{h_T^*}{\|h_T^*\|_{\mathcal{PW}_\pi^\infty}}.$$

Then we have

$$\|h_{T_2} - h_{T_1}\|_{\mathcal{PW}_\pi^\infty} = \frac{\delta}{2} < \delta$$

and from (40) it follows that

$$\Theta_N(h_{T_2}) \leq C_7.$$

For $k \in \mathbb{Z}$ we have, due to the linearity of c_k in the first argument,

$$\begin{aligned}\frac{\delta c_k(h_T^*,\phi^*)}{2\|h_T^*\|_{\mathcal{PW}_\pi^\infty}} &= c_k\left(\frac{\delta h_T^*}{2\|h_T^*\|_{\mathcal{PW}_\pi^\infty}},\phi^*\right)\\ &= c_k\left(\frac{\delta h_T^*}{2\|h_T^*\|_{\mathcal{PW}_\pi^\infty}} + h_{T_1} - h_{T_1},\phi^*\right)\\ &= c_k(h_{T_2},\phi^*) - c_k(h_{T_1},\phi^*),\end{aligned}$$

and consequently

$$\frac{\delta}{2\|h_T^*\|_{\mathcal{PW}_\pi^\infty}}|c_k(h_T^*,\phi^*)| \leq |c_k(h_{T_2},\phi^*)| + |c_k(h_{T_1},\phi^*)|.$$

It follows that

$$\begin{aligned}\frac{\delta}{2\|h_T^*\|_{\mathcal{PW}_\pi^\infty}} \sum_{k=-N}^{N} |c_k(h_T^*, \phi^*)| &\leq \sum_{k=-N}^{N} |c_k(h_{T_2}, \phi^*)| + \sum_{k=-N}^{N} |c_k(h_{T_1}, \phi^*)| \\ &= \Theta_N(h_{T_2}) + \Theta_N(h_{T_1}) \\ &\leq 2C_7,\end{aligned}$$

which shows that

$$\sum_{k=-\infty}^{\infty} |c_k(h_T^*, \phi^*)| \leq \frac{4C_7\|h_T^*\|_{\mathcal{PW}_\pi^\infty}}{\delta} < \infty.$$

This is a contradiction to (38). Thus our assumption was wrong, i.e., (39) can only hold for a set $\mathcal{K}$ of the first category. □

We give the remaining proof of Lemma 2.

Proof of Lemma 2. Let all assumptions of the lemma be satisfied, and let $t \in \mathbb{R}$ be arbitrary but fixed. For $f \in \mathcal{I}$ and $N \in \mathbb{N}$ let

$$\Phi_N(f) = \sup_{\|h_T\|_{\mathcal{PW}_\pi^\infty} \leq 1} \left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right|.$$

According to the uniform boundedness theorem [33, p. 98], we have $\sup_{N\in\mathbb{N}} \Phi_N(f) < \infty$ for all $f \in \mathcal{I}$. $\{\Phi_N\}_{N\in\mathbb{N}}$ is a sequence of continuous convex functionals. From the generalized uniform boundedness theorem (see Theorem 7 in the appendix), it follows that there exist an $f_1 \in \mathcal{I}$, a $\delta > 0$, and a constant C_8 such that $\Phi_N(f) \leq C_8$ for all $f \in \mathcal{I}$ with $\|f - f_1\|_{\mathcal{I}} < \delta$. Since Φ_N is convex and positive homogeneous, i.e., satisfies $\Phi_N(\lambda f) = |\lambda|\Phi_N(f)$, it follows that there exists a constant C_9 such that $\Phi_N(f) \leq C_9\|f\|_{\mathcal{I}}$ for all $f \in \mathcal{I}$. Since

$$\begin{aligned}\left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| &\leq \Phi_N(f)\|h_T\|_{\mathcal{PW}_\pi^\infty} \\ &\leq \sup_{N\in\mathbb{N}} \Phi_N(f)\|h_T\|_{\mathcal{PW}_\pi^\infty},\end{aligned}$$

it follows that

$$\left| \sum_{k=-N}^{N} f(k) h_T(t-k) \right| \leq C_9\|f\|_{\mathcal{I}}\|h_T\|_{\mathcal{PW}_\pi^\infty}$$

for all $f \in \mathcal{I}$ and $h_T \in \mathcal{PW}_\pi^\infty$. □

XI. Conclusion

In the present paper we analyzed the existence of the convolution sum for signals $f \in \mathcal{C}$ and functions $h_T \in \mathcal{PW}_\pi^\infty$. We proved that, in general, the convolution sum does not exist, even in a distributional setting. In contrast, the convolution integral is always well-defined. This result shows that the usual operation of multiplying a signal with a Dirac comb and subsequent convolution cannot be legitimized by distribution theory.

A similar result was shown in [34] for the downsampling of bounded bandlimited signals. The formal application of distribution theory suggests that downsampling of bounded bandlimited signals is a completely benign operation that creates no problems. However, as shown in [34], this is not the case.

Further, applying distributions in the analysis and modeling of non-linear systems can be problematic. Since distributions are continuous linear functionals on a suitable test function space, and non-linear effects are in general not compatible with that structure, distribution theory is hardly applicable to non-linear problems. Such a theory needs to be further developed in the future.

Even more, the convolution has a structure that is not directly compatible with the linear structure of distributions. Although the convolution is a bilinear operation, i.e., linear in each of its arguments, a linear variation of both arguments at the same time produces a non-linear effect. This dependence is another reason why a convolution can in general not be defined for distributions.

Appendix

Uniform Boundedness Theorem

A key result in functional analysis is the uniform boundedness theorem [35, Theorem 16.2, p. 45], which we will state next. Let $\mathcal{X}$ be a Banach space. By

$$\mathcal{U}_{\tilde{\epsilon}}(\tilde{x}) = \{x \in \mathcal{X} \colon \|x - \tilde{x}\|_{\mathcal{X}} < \tilde{\epsilon}\}$$

we denote the open ball at $\tilde{x}$ with radius $\tilde{\epsilon}$.

Theorem 7 (Generalized Uniform Boundedness Theorem)**.** *Let $\mathcal{X}$ be a Banach space and $\underline{\mathcal{X}}$ a set of the second category in $\mathcal{X}$. Further, let $\mathcal{F}$ be a set of continuous functions mapping from $\mathcal{X}$ into $\mathbb{R}$, and satisfying*

$$\sup_{F\in\mathcal{F}} F(\underline{x}) < \infty \tag{41}$$

for all $\underline{x} \in \underline{\mathcal{X}}$. Then there exists an open ball $\mathcal{U}_{\tilde{\epsilon}}(\tilde{x})$ in $\mathcal{X}$ and a constant $C < \infty$ such that

$$F(x) \leq C$$

for all $x \in \mathcal{U}_{\tilde{\epsilon}}(\tilde{x})$ and all $F \in \mathcal{F}$.

Theorem 7 is slightly more general than Theorem 16.2 in [35]. We require (41) to hold only for a set of the second category instead of the whole space. Nevertheless, the proof of Theorem 7 is similar to the proof in [35, Theorem 16.2, p. 45].

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