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Turing Computability of Fourier Transforms of Bandlimited and Discrete Signals

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***Abstract*—The Fourier transform is an important operation in signal processing. However, its exact computation on digital computers can be problematic. In this paper we consider the computability of the Fourier transform and the discrete-time Fourier transform (DTFT). We construct a computable bandlimited absolutely integrable signal that has a continuous Fourier transform, which is, however, not Turing computable. Further, we also construct a computable sequence such that the DTFT is not Turing computable. Turing computability models what is theoretically implementable on a digital computer. Hence, our result shows that the Fourier transform of certain signals cannot be computed on digital hardware of any kind, including CPUs, FPGAs, and DSPs. This also implies that there is no symmetry between the time and frequency domain with respect to computability. Therefore, numerical approaches which employ the frequency domain representation of a signal (like calculating the convolution by performing a multiplication in the frequency domain) can be problematic. Interestingly, an idealized analog machine can compute the Fourier transform. However, it is unclear whether and how this theoretical superiority of the analog machine can be translated into practice. Further, we show that it is not possible to find an algorithm that can always decide for a given signal whether the Fourier transform is computable or not.**

***Index Terms*—Fourier transform, discrete-time Fourier transform, algorithmic decision, Turing computability, frequency domain**

I. Introduction

THE Fourier transform and the discrete-time Fourier transform are two important operations in signal processing [2]–[5]. Using those transforms we can interpret signals in terms of their frequency composition. A useful property of the Fourier transform is that it transforms a convolution in the time domain into a multiplication in the frequency domain. For this reason, the output of a linear time-invariant (LTI) system can be easily determined in the frequency domain by multiplying the system input with the transfer function of the LTI system.

For practical applications, it is essential that we can compute the Fourier transform on a digital computer [6]. But even though theorems like the convolution theorem play an important role in signal processing [5], the computability of the Fourier transform has not gotten much attention. In this paper we will study the computability of the Fourier transform and the discrete-time Fourier transform (DTFT). The proper framework to treat this question is Turing computability. A Turing machine is an abstract device that manipulates symbols on a strip of tape according to certain rules [7]–[10]. Although the concept is very simple, a Turing machine is capable of simulating any given algorithm. A Turing machine has no limitations in terms of memory or computing time, and hence provides a theoretical model that describes the fundamental limits of any practically realizable digital computer. This implies that anything that is not computable on a Turing machine, cannot be computed on any digital hardware, including CPUs, FPGAs, and DSPs.

We will show that there are signals for which no Turing machine exists, and hence no algorithm that can compute the Fourier transform. The same holds true for the discrete-time Fourier transform (DTFT).

As for the Fourier transform, we will construct an absolutely integrable bandlimited signal f_*, which itself is computable as a continuous signal, such that its Fourier transform $\hat{f}_*$ is continuous but not Turing computable, because $\hat{f}_*(0)$ is not computable. The signal f_* has further interesting properties. The $L^p(\mathbb{R})$-norms of f_* are computable for all computable $1 < p \leq \infty$, and, in particular, the energy, i.e., the $L^2(\mathbb{R})$-norm is computable. We will give a precise definition of what we mean by "computable" in Section III.

In [9, p. 110, Th. 4] a positive result about the computability of the Fourier transform was given for certain $L^p(\mathbb{R})$ spaces. To obtain this result, the fact was used that the Fourier transform is a bounded operator from $L^p(\mathbb{R})$ into $L^q(\mathbb{R})$ if $p \leq 2$ and q is the conjugate index of p, satisfying $1/p + 1/q = 1$. Further, Type-2 computability of the Fourier transform for $L^p(\mathbb{R})$ was studied in [11]. In [12], without a proof, a computable continuous signal was stated that has a non-computable Fourier transform. The construction and the properties of the signal f_* in the present paper are different from the signal given in [12]. In particular, f_* is bandlimited, which enables us to use Shannon's theory of sampling series to compute its L^p-norms. Further, our approach immediately gives us a finite Shannon sampling series approximation for f_*, with effective control of the approximation error. Such an effective control of the approximation error is not possible in the frequency domain, i.e., for the approximation of $\hat{f}_*$, because $\hat{f}_*(0)$ is not computable.

As for the DTFT, we will construct a sequence that is computable in ℓ^p, $1 < p < \infty$, such that its DTFT is not

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computable on all dyadic grids. Although the convergence behavior of the Fourier series is important and a well-studied topic in classical analysis [13], questions of computability have not caught much attention. The convergence of Fourier series for computable Lebesgue integrable signals was studied in [14], and it has been shown that the set of L^1-computable signals, whose Fourier series diverges almost everywhere, is big in a certain sense. In [12] a computable signal $f(t)$ was given, for which the Fourier series converges uniformly, but the convergence is not effective for $t = 0$. This result does not imply ours, because it only shows that the Fourier series of f does not converge effectively. We instead show that the DTFT signal itself is not computable, no matter what procedure is used to calculate it.

Further, we investigate the problem whether it is possible to find an algorithm that can decide for a given signal if its Fourier transform is computable. We will show that such an algorithm cannot exist. This fact has consequences, e.g., for computer aided control system design (CAD), where such an algorithm would be necessary in order that problematic signals be avoided.

Other works that treat computability in the context of signal processing are [15], [16]. In [15] downsampling and the bandlimited interpolation have been studied with respect to computability, and in [16] the decidability of the uniform convergence of the Fourier series was analyzed.

II. Notation

By $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$, we denote the usual spaces of p-th power summable sequences $x = \{x(k)\}_{k\in\mathbb{Z}}$ with the norm $\|x\|_{\ell^p} = (\sum_{k=-\infty}^{\infty}|x(k)|^p)^{1/p}$. $\ell^p_+(\mathbb{Z})$ denotes the set of sequences $\{x(k)\}_{k\in\mathbb{Z}}$ in $\ell^p(\mathbb{Z})$ that vanish for $k < 0$, i.e., satisfy $x(k) = 0$ for all $k < 0$.

In this paper we will use both the terms function and signal interchangeably. By C we denote the space of all continuous functions on $\mathbb{R}$ that vanish at infinity, equipped with the norm $\|f\|_C = \max_{t\in\mathbb{R}}|f(t)|$. For $\Omega \subseteq \mathbb{R}$, let $L^p(\Omega)$, $1 \leq p < \infty$, be the space of all measurable p-th power Lebesgue integrable functions on Ω with the usual norm $\|\cdot\|_p$, and $L^\infty(\Omega)$ the space of all functions for which the essential supremum norm $\|\cdot\|_\infty$ is finite. For $0 < \sigma < \infty$ and $1 \leq p \leq \infty$, we denote by $\mathcal{B}^p_\sigma$ the Bernstein space of all functions of exponential type at most σ, whose restriction to the real line is in $L^p(\mathbb{R})$ [17, p. 49]. The norm for $\mathcal{B}^p_\sigma$ is given by the L^p-norm on the real line. A function in $\mathcal{B}^p_\sigma$ is called bandlimited to σ. $\mathcal{B}^2_\sigma$ is the frequently used space of bandlimited functions with bandwidth σ and finite energy. We have $\mathcal{B}^p_\sigma \subset \mathcal{B}^r_\sigma$ for all $1 \leq p \leq r \leq \infty$ [17, p. 49, Lemma 6.6]. $\mathcal{B}^\infty_{\sigma,0}$ denotes the space of all functions in $\mathcal{B}^\infty_\sigma$ that vanish at infinity.

By $\partial\mathbb{D}$ we denote the boundary of the unit disk, i.e., the unit circle, and $C(\partial\mathbb{D})$ denotes the set of all continuous functions on $\partial\mathbb{D}$. We equip $C(\partial\mathbb{D})$ with the norm $\|f\|_C = \max_{\omega\in[0,2\pi)}|f(\mathrm{e}^{i\omega})|$. The Wiener algebra $\mathcal{W}$ is the space of all functions in $C(\partial\mathbb{D})$ with an absolutely convergent Fourier series.

III. Computability

The theory of computability is a well-established field in computer science [7]–[10], [18]. However, since computability is not widely known in the signal processing community, we describe some of the key concepts in this section. For a more detailed treatment of the topic, see for example [9], [10], [18], [19].

In order to study the question of computability, we employ the concept of Turing computability. A Turing machine is an abstract device that manipulates symbols on a strip of tape according to certain rules [7], [8], [10], [18]. Despite their simplicity, Turing machines are capable of simulating any given algorithm. Further, Turing machines are equivalent to other concepts of computability, such as those defined by general recursive functions, Minksy register machines, or λ-calculus. Since Turing machines have no limitations in terms of memory or computing time, they provide a theoretical model that describes the fundamental limits of any practically realizable digital computer.

It is important to distinguish Turing computability from complexity theory, another topic in computer science. Complexity theory deals with the question how efficiently a problem can be solved, and analyzes how the computation time of a given algorithm scales with the size of the input data. Thus, the goal of complexity theory is different to the goal in Turing computability where the fundamental limits of computability are explored, without consideration of complexity issues. Further, complexity theory operates in a discrete and finite setting. However, in the modeling of many real world problems, continuous functions are used, e.g., bandlimited functions that have an infinite duration. In principle, computability theory can make statements about the computability of such objects.

Alan Turing introduced the concept of a computable real number in [7], [8]. A sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ is called a computable sequence if there exist recursive functions a, b, s from $\mathbb{N}$ to $\mathbb{N}$ such that $b(n) \neq 0$ for all $n \in \mathbb{N}$ and

$$r_n = (-1)^{s(n)}\frac{a(n)}{b(n)}, \quad n \in \mathbb{N}.$$

A recursive function is a function, mapping natural numbers into natural numbers, that is built of simple computable functions and recursions [20]. Recursive functions are computable by a Turing machine. A real number x is said to be computable if there exists a computable sequence of rational numbers $\{r_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$ such that $|x - r_{\xi(n)}| < 2^{-n}$ for all $n \in \mathbb{N}$. By $\mathbb{R}_c$ we denote the set of computable real numbers and by $\mathbb{C}_c = \mathbb{R}_c + i\mathbb{R}_c$ the set of computable complex numbers. $\mathbb{R}_c$ is a field; i.e. finite sums, differences, products, and quotients of computable numbers are computable. Note that commonly used constants like e and π are computable. A non-computable real number was, for example, constructed in [21].

A sequence $\{x(k)\}_{k\in\mathbb{Z}}$ in ℓ^p, $p \in [1,\infty)\cap\mathbb{R}_c$ is called computable in ℓ^p if: 1) every number $x(k)$, $k \in \mathbb{Z}$, is computable, and 2) there exist a computable sequence $\{x_N\}_{N\in\mathbb{N}} \subset \ell^p$, where each x_N has only finitely many non-zero elements, all of which are computable as real numbers, and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$, such that for all $n \in \mathbb{N}$ we have

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$\|x - x_{\xi(n)}\|_{\ell^p} \leq 2^{-n}$. By $\mathcal{C}\ell^p$ we denote the set of all sequences that are computable in ℓ^p. Similarly, we define $\mathcal{C}\ell^p_+(\mathbb{Z})$ as the set of all sequences in $\ell^p_+(\mathbb{Z})$ that are computable in ℓ^p.

There are several—not equivalent—definitions of computable functions, most notably, Turing computable functions, Markov computable functions, and Banach–Mazur computable functions [19]. For us, the connection between Banach–Mazur computability and Turing computability is important. Any function that is Turing computable is always Banach–Mazur computable. Conversely, any function that is not Banach–Mazur computable cannot be Turing computable. A function $f\colon \mathbb{R} \to \mathbb{R}$ is called Banach–Mazur computable if f maps any given computable sequence $\{x_n\}_{n\in\mathbb{N}}$ of real numbers into a computable sequence $\{f(x_n)\}_{n\in\mathbb{N}}$ of real numbers. It follows that a function that is computable with respect to any of the above definitions has the property that it maps computable numbers into computable numbers. This property is therefore a necessary condition for computability. Usual functions like $\sin$, sinc, $\log$, and $\exp$ are Turing computable, and finite sums of computable functions are Turing computable [9]. We will further use the important fact that every computable real function is continuous on $\mathbb{R}_c$ [19]. For a more detailed treatment of computability, see for example [9], [10], [18], [19], and for an example of a non-computable function [22].

We call a function f elementary computable if there exists a natural number N and computable numbers $\{\alpha_k\}_{k=-N}^{N}$ such that

$$f(t) = \sum_{k=-N}^{N} \alpha_k \frac{\sin(\pi(t-k))}{\pi(t-k)}. \tag{1}$$

Note that every elementary computable function f is a finite sum of Turing computable functions, and hence, Turing computable. As a consequence, for every $t \in \mathbb{R}_c$, the number $f(t)$ is computable. Further, the sum of finitely many elementary computable functions is computable, as well as the product of an elementary computable function with a computable number $\lambda \in \mathbb{C}_c$. Hence, the set of elementary computable functions is closed with respect to the operations addition and multiplication with a scalar. Further, for every elementary computable function f, the norm $\|f\|_{\mathcal{B}^p_\pi}$, $p \in (1,\infty] \cap \mathbb{R}_c$, is computable.

A function in $f \in \mathcal{B}^p_\pi$, $p \in [1,\infty) \cap \mathbb{R}_c$, is called computable in $\mathcal{B}^p_\pi$ if there exists a computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$, such that $\|f - f_{\xi(n)}\|_{\mathcal{B}^p_\pi} \leq 2^{-n}$ for all $n \in \mathbb{N}$. By $\mathcal{CB}^p_\pi$ we denote the set of all functions that are computable in $\mathcal{B}^p_\pi$. Note that $\mathcal{CB}^p_\pi$ has a linear structure. We can approximate every function $f \in \mathcal{CB}^\infty_\pi$ by an elementary computable function, where we have an "effective" control of the approximation error. This control of the error is illustrated in Fig. 1.

Similarly, we define the set $\mathcal{CB}^\infty_{\pi,0}$ of all functions in $\mathcal{B}^\infty_{\pi,0}$ that are computable in $\mathcal{B}^\infty_{\pi,0}$. A function in $f \in \mathcal{B}^\infty_{\pi,0}$ is called computable in $\mathcal{B}^\infty_{\pi,0}$ if there exists a computable sequence of elementary computable functions $\{f_n\}_{n\in\mathbb{N}}$ and a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$, such that $\|f - f_{\xi(n)}\|_{\mathcal{B}^\infty_{\pi,0}} \leq 2^{-n}$ for all $n \in \mathbb{N}$. Since for every elementary computable function

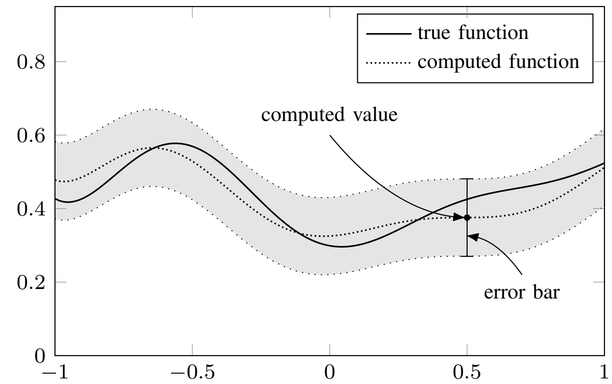

Fig. 1. For a computable function we can always determine an error bar and be sure that the true value lies within the specified error range.

f_n, the norm $\|f_n\|_{\mathcal{B}^\infty_{\pi,0}}$ is computable, it follows immediately from the inequality

$$\left| \|f\|_{\mathcal{B}^\infty_{\pi,0}} - \|f_n\|_{\mathcal{B}^\infty_{\pi,0}} \right| \leq \|f - f_n\|_{\mathcal{B}^\infty_{\pi,0}},$$

that the norm $\|f\|_{\mathcal{B}^\infty_{\pi,0}}$, i.e., the maximum of f, is computable for all $f \in \mathcal{CB}^\infty_{\pi,0}$. See also [9, pp. 40].

In order that the above definition of a computable function in $\mathcal{B}^\infty_{\pi,0}$ is meaningful, it is necessary that each $f \in \mathcal{B}^\infty_{\pi,0}$ can be approximated in a classical sense by a linear combination of shifted sinc-functions. This is assured by the next fact.

Fact 1. *Let $f \in \mathcal{B}^\infty_{\pi,0}$. For every $\epsilon > 0$ there exists an $N \in \mathbb{N}$ and numbers $\{c_k\}_{k=-N}^{N}$ such that*

$$\left\| f - \sum_{k=-N}^{N} c_k \frac{\sin(\pi(t-k))}{\pi(t-k)} \right\|_{\mathcal{B}^\infty_{\pi,0}} < \epsilon.$$

A set $\mathcal{A} \subseteq \mathbb{N}$ is called recursively enumerable if $\mathcal{A} = \emptyset$ or $\mathcal{A}$ is the range of a recursive function. A set $\mathcal{A} \subseteq \mathbb{N}$ is called recursive if both $\mathcal{A}$ and its complement $\mathbb{N} \setminus \mathcal{A}$ are recursively enumerable. The fact that there exist sets which are recursively enumerable but not recursive will be important for us [9, p. 7, Proposition A], [20, p. 18].

The following lemma [9, p. 20, Corollary 2b], in which a non-computable number is constructed from a recursively enumerable nonrecursive set, is essential for us.

Lemma 1. *Let $\mathcal{A} \subset \mathbb{N}$ be a recursively enumerable nonrecursive set, and $\phi_{\mathcal{A}}\colon \mathbb{N} \to \mathcal{A}$ a recursive enumeration of the elements of $\mathcal{A}$, where $\phi_{\mathcal{A}}$ is a one-to-one function, i.e., for every element $k \in \mathcal{A}$ there exists exactly one $N_k \in \mathbb{N}$ with $\phi_{\mathcal{A}}(N_k) = k$. Then the number $\sum_{N=1}^{\infty} 2^{-\phi_{\mathcal{A}}(N)}$ is not computable.*

IV. Fourier Optics

The field of Fourier optics is a well-established discipline in physics and optics that is older than Turing's theory of computability and digital computers [23]. The $2f$ architecture in Fourier optics is an optical setup, in which a lens is used to perform the Fourier transform. This setup can be seen as an analog machine for computing the Fourier transform of

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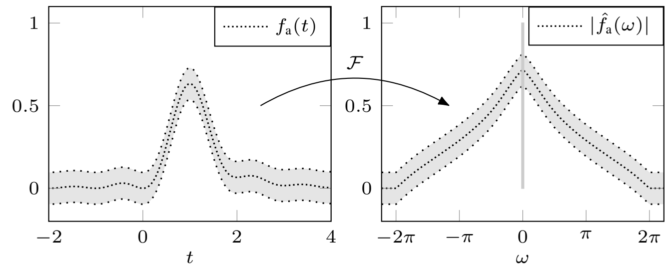

Fig. 2. Illustration of the non-computability of the Fourier transform. In the left panel we see an approximation $f_a(t)$ of the computable time domain function $f_*(t)$ (the true values lay within the gray shaded area). In the right panel we see the absolute value of the Fourier transform $|\hat{f}_a(\omega)| = |(\mathcal{F}f_a)(\omega)|$. The values of $\hat{f}_a(\omega)$ are computable for all $\omega \in \mathbb{R}_c \setminus \{0\}$, but not for $\omega = 0$ (indicated by the gray vertical bar).

a bandlimited function [24]. In theory, the Fourier transform computed by such a system is perfect, but in any practical realization there are always imperfections, such as misalignment and noise, that limit the precision with which the Fourier transform can be computed.

There are studies that try to construct and analyze computing machines based on analog components [25]–[31]. In [25], [26], the computational power of networks of spiking neurons has been analyzed. Further, in [27], [28], DNA based Turing machines were analyzed, and chemical implementations were investigated in [29]–[31]. However, the authors of the above publications do not seek for analog computers, instead they use the analog components to construct a digital computer, i.e., a Turing machine.

The rationale behind these approaches is that being able to implement a Turing machine with the analog components shows the ultimate potential of the analog device. However, this is not necessarily the case. If we consider the Turing machine as an idealized model of a digital computer with unlimited memory and computation time, and the $2f$ architecture in Fourier optics as an idealized analog device without noise and other imperfections, our result shows that in certain situations an idealized analog machine is more powerful than an idealized digital machine: The idealized analog machine is capable of computing the Fourier transform, while the Fourier transform is not Turing computable. However, it is unclear so far whether this theoretical superiority can be translated into practice.

V. Fourier Transform

In this section we study the computability of the Fourier transform of certain bandlimited functions. We construct a continuous bandlimited function $f_* \in \mathcal{B}_{2\pi}^1$ that is computable as an element of $\mathcal{B}_{2\pi}^p$ for all $1 < p < \infty$, $p \in \mathbb{R}_c$, and as an element of $\mathcal{B}_{2\pi,0}^\infty$, such that the Fourier transform $\hat{f}_*$ is not a Turing computable function because $\hat{f}_*(0) \notin \mathbb{C}_c$. Note that $f_*(t) \in \mathbb{C}_c$ for all $t \in \mathbb{R}_c$, i.e., $f_*(t)$ is computable for all computable t, but $\hat{f}_*(0)$ is not computable. This situation is illustrated in Fig. 2.

Theorem 1. *We construct a function $f_* \in \mathcal{B}_{2\pi}^1$ that is computable as an element of $\mathcal{B}_{2\pi,0}^\infty$ and as an element of $\mathcal{B}_{2\pi}^p$ for all $p \in (1, \infty) \cap \mathbb{R}_c$, such that f_* has a continuous Fourier transform $\hat{f}_*$ that is not computable in C, because $\hat{f}_*(0) \notin \mathbb{C}_c$. Further, the function f_* is constructed such that $\hat{f}_*(\omega) \in \mathbb{C}_c$ for all $\omega \in \mathbb{R}_c \setminus \{0\}$.*

Remark 1. The result from Theorem 1 is interesting, because we have $\hat{f}_*(0) \notin \mathbb{C}_c$, but $\hat{f}_*(\omega) \in \mathbb{C}_c$ for all $\omega \in \mathbb{R}_c \setminus \{0\}$ and

$$\lim_{\omega \to 0} \hat{f}_*(\omega) = \hat{f}_*(0), \tag{2}$$

due to the continuity of $\hat{f}_*$. At a fist glance this might seem surprising. The explanation is that the convergence in (2) is not effective.

Remark 2. Since f_* in our paper is bandlimited, it follows that there exists a simple series expansion of the Fourier transform:

$$\hat{f}_*(\omega) = \frac{1}{2} \sum_{k=-\infty}^{\infty} f_*\left(-\frac{k}{2}\right) \mathrm{e}^{i\omega k/2}, \quad \omega \in (-2\pi, 2\pi). \tag{3}$$

Thus, for the calculation of $\hat{f}_*$ we have, in addition to the Fourier integral, the series expression (3), which only requires the samples of f_*.

Remark 3. Since $f_* \in \mathcal{B}_{2\pi}^1$, we have

$$\sum_{k=-\infty}^{\infty} \left| f_*\left(\frac{k}{2}\right) \right| < \infty,$$

according to Nikol'skiĭ's inequality [17, p. 49, Th. 6.8]. Hence, it follows that the series in

$$\hat{f}_*(\omega) = \frac{1}{2} \sum_{k=-\infty}^{\infty} f_*\left(-\frac{k}{2}\right) \mathrm{e}^{i\omega k/2}, \quad \omega \in (-2\pi, 2\pi),$$

converges absolutely. Further, we have

$$\left| \hat{f}_*(\omega) - \frac{1}{2} \sum_{k=-N}^{N} f_*\left(-\frac{k}{2}\right) \mathrm{e}^{i\omega k/2} \right| \leq \frac{1}{2} \sum_{|k| \geq N} \left| f_*\left(-\frac{k}{2}\right) \right|.$$

Since $f_*(-k/2) \in \mathbb{R}_c$, $k \in \mathbb{Z}$, it follows that

$$\frac{1}{2} \sum_{k=-N}^{N} f_*\left(-\frac{k}{2}\right) \mathrm{e}^{i\omega k/2}$$

is a computable trigonometric polynomial. Theorem 1 implies that this computable sequence of computable trigonometric polynomials does not converge effectively to $\hat{f}_*$ in the maximum norm, because $\hat{f}_*(0)$ is not computable. Since $\hat{f}_*(0)$ is not computable, it also follows that is impossible to find any other computable sequence of computable trigonometric polynomials that converges effectively to $\hat{f}_*$ in the maximum norm.

For the proof of Theorem 1, we need the following two elementary lemmas, the proofs of which are given in Appendices B and C, respectively. It is important that the constants on the right hand sides of the inequalities in Lemmas 2 and 3 are computable. The usual upper bounds that exist, see for example [32, pp. 182–192], contain only general constants, and therefore are useless for us.

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Lemma 2. *For all $N \in \mathbb{N}$ and all $\omega \in \mathbb{R}$ we have*

$$\left| \sum_{k=1}^{N} \frac{1}{k} \sin(k\omega) \right| < \pi.$$

Lemma 3. *For all $N \in \mathbb{N}$ and all $0 < \delta \leq 1/2$ we have*

$$\left| \sum_{k=1}^{N} \frac{1}{k} \cos(k\omega) \right| \leq \log\left(\frac{1}{\delta}\right) + 2 + 2\pi$$

for all $\omega \in \mathbb{R}$ satisfying $|\omega - k2\pi| \geq \delta$ for all $k \in \mathbb{Z}$.

Proof of Theorem 1. For $N \in \mathbb{N}$, let

$$g_N(t) = \sum_{k=1}^{N} \frac{1}{k} \left(\frac{\sin(\pi(t-k))}{\pi(t-k)} \right)^2, \quad t \in \mathbb{R}.$$

We have

$$\begin{aligned} \int_{-\infty}^{\infty} |g_N(t)| \, \mathrm{d}t &= \int_{-\infty}^{\infty} g_N(t) \, \mathrm{d}t \\ &= \sum_{k=1}^{N} \frac{1}{k} \int_{-\infty}^{\infty} \left(\frac{\sin(\pi(t-k))}{\pi(t-k)} \right)^2 \mathrm{d}t \\ &= \sum_{k=1}^{N} \frac{1}{k}, \end{aligned} \tag{4}$$

where we used that

$$\begin{aligned} \int_{-\infty}^{\infty} \left(\frac{\sin(\pi(t-k))}{\pi(t-k)} \right)^2 \mathrm{d}t &= \int_{-\infty}^{\infty} \left(\frac{\sin(\pi t)}{\pi t} \right)^2 \mathrm{d}t \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} 1 \, \mathrm{d}\omega \\ &= 1, \end{aligned} \tag{5}$$

according to Plancherel's identity. Thus, we see that $g_N \in \mathcal{B}_{2\pi}^1$. Since $\mathcal{B}_{2\pi}^1 \subset \mathcal{B}_{2\pi}^p$ for all $1 \leq p \leq \infty$ [17, p. 49, Lemma 6.6], it follows that $g_N \in \mathcal{B}_{2\pi}^p$ for all $1 \leq p \leq \infty$. Further, we have

$$\hat{g}_N(\omega) = \left(\sum_{k=1}^{N} \frac{1}{k} \mathrm{e}^{-i\omega k} \right) \hat{q}(\omega),$$

where

$$\hat{q}(\omega) = \int_{-\infty}^{\infty} \left(\frac{\sin(\pi t)}{\pi t} \right)^2 \mathrm{e}^{-i\omega t} \, \mathrm{d}t = \begin{cases} 1 - \frac{|\omega|}{2\pi}, & |\omega| \leq 2\pi, \\ 0, & |\omega| > 2\pi. \end{cases}$$

Since $\hat{q}(0) = 1$, it follows that

$$\begin{aligned} \hat{g}_N(0) &= \sum_{k=1}^{N} \frac{1}{k} \\ &\geq \sum_{k=1}^{N} \int_{k}^{k+1} \frac{1}{\tau} \, \mathrm{d}\tau \\ &= \int_{1}^{N+1} \frac{1}{\tau} \, \mathrm{d}\tau \\ &= \log(N+1). \end{aligned} \tag{6}$$

It can be shown that g_N is computable in $\mathcal{B}_{2\pi}^p$ and that the norm

$$\|g_N\|_p = \left(\int_{-\infty}^{\infty} |g_N(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}}$$

is computable for $1 < p < \infty$, $p \in \mathbb{R}_c$. We prove the second fact in Appendix D.

Next, we will derive upper bounds for the $L^p(\mathbb{R})$-norms of g_N for $1 < p < \infty$. For $1 < p < \infty$, we have, according to the Plancherel–Pólya inequality [33, p. 152, Theorem 3], that

$$\left(\int_{-\infty}^{\infty} |g_N(t)|^p \, \mathrm{d}t \right)^{\frac{1}{p}} \leq C(p) \left(\sum_{k=-\infty}^{\infty} \left| g_N\left(\frac{k}{2}\right) \right|^p \right)^{\frac{1}{p}} \tag{7}$$

where

$$C(p) = \frac{(1+\pi)}{2^{1/p}} \cdot \begin{cases} \cot(\frac{\pi(p-1)}{2p}), & 1 < p < 2, \\ \tan(\frac{\pi(p-1)}{2p}), & 2 \leq p < \infty, \end{cases}$$

is a constant that only depends on p. The value of the constant $C(p)$ is derived in Appendix A. We have $C(p) \in \mathbb{R}_c$ for all $p \in (1, \infty) \cap \mathbb{R}_c$. For $k \in \mathbb{Z}$ we have

$$g_N(k) = \begin{cases} \frac{1}{k}, & 1 \leq k \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

Let

$$a_N(l) = g_N\left(l + \frac{1}{2}\right), \quad l \in \mathbb{Z},$$

and

$$b(l) = \left(\frac{\sin(\pi(l+\frac{1}{2}))}{\pi(l+\frac{1}{2})} \right)^2, \quad l \in \mathbb{Z}.$$

Then we have

$$a_N(l) = \sum_{k=1}^{N} \frac{1}{k} \left(\frac{\sin(\pi(l+\frac{1}{2}-k))}{\pi(l+\frac{1}{2}-k)} \right)^2 = \sum_{k=1}^{N} \frac{1}{k} b(l-k).$$

Since

$$\begin{aligned} \|b\|_{\ell^1} &= \sum_{l=-\infty}^{\infty} |b(l)| = \sum_{l=-\infty}^{\infty} b(l) \\ &= \sum_{l=-\infty}^{\infty} \left(\frac{\sin(\pi(l+\frac{1}{2}))}{\pi(l+\frac{1}{2})} \right)^2 \\ &= \int_{-\infty}^{\infty} \left(\frac{\sin(\pi(t+\frac{1}{2}))}{\pi(t+\frac{1}{2})} \right)^2 \mathrm{d}t \\ &= \int_{-\infty}^{\infty} \left(\frac{\sin(\pi t)}{\pi t} \right)^2 \mathrm{d}t \\ &= 1, \end{aligned}$$

where the third line follows from [17, p. 50, Th. 6.11] and the last line from (5). Using Young's convolution inequality we obtain

$$\|a_N\|_{\ell^p} \leq \|b\|_{\ell^1} \left(\sum_{k=1}^{N} \frac{1}{k^p} \right)^{\frac{1}{p}} = \left(\sum_{k=1}^{N} \frac{1}{k^p} \right)^{\frac{1}{p}}. \tag{8}$$

For $k \in \mathbb{Z}$ we have

$$g_N(k) = \begin{cases} \frac{1}{k}, & 1 \leq k \leq N, \\ 0, & \text{otherwise.} \end{cases}$$

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It follows that

$$\begin{aligned}\sum_{k=-\infty}^{\infty} \left| g_N\left(\frac{k}{2}\right)\right|^p &= \sum_{k=-\infty}^{\infty} |g_N(k)|^p + \sum_{k=-\infty}^{\infty} \left| g_N\left(k+\frac{1}{2}\right)\right|^p \\ &= \sum_{k=1}^{N} \frac{1}{k^p} + \sum_{k=-\infty}^{\infty} |a_N(k)|^p \\ &\leq 2\sum_{k=1}^{N}\frac{1}{k^p}, \end{aligned} \tag{9}$$

where we used (8) in the last line. Combining (7) and (9), we see that

$$\begin{aligned}\|g_N\|_p &\leq C(p)\left(2\sum_{k=1}^{N}\frac{1}{k^p}\right)^{\frac{1}{p}} \\ &\leq 2^{\frac{1}{p}} C(p)\left(1+\sum_{k=2}^{N}\int_{k-1}^{k}\frac{1}{\tau^p}\,\mathrm{d}\tau\right)^{\frac{1}{p}} \\ &= 2^{\frac{1}{p}} C(p)\left(1+\int_{1}^{N}\frac{1}{\tau^p}\,\mathrm{d}\tau\right)^{\frac{1}{p}} \\ &= 2^{\frac{1}{p}} C(p)\left(1+\frac{1}{p-1}-\frac{1}{(p-1)N^{p-1}}\right)^{\frac{1}{p}} \\ &= 2^{\frac{1}{p}} C(p)\left(1+\frac{1}{p-1}\right)^{\frac{1}{p}} =: C_1(p), \end{aligned} \tag{10}$$

where, for all $p \in (1,\infty)\cap\mathbb{R}_c$, the constant $C_1(p)$ is a computable number.

For $N \in \mathbb{N}$, let

$$h_N(t) = \frac{g_N(t)}{\hat{g}_N(0)}, \quad t \in \mathbb{R}.$$

Since g_N is computable in $\mathcal{B}_{2\pi}^p$, $p \in (1,\infty)\cap\mathbb{R}_c$, and $\hat{g}_N(0)$ is a computable number, it follows that h_N is computable in $\mathcal{B}_{2\pi}^p$. We further have

$$\|h_N\|_p \leq \frac{C_1(p)}{\hat{g}_N(0)} \leq \frac{C_1(p)}{\log(N+1)}, \tag{11}$$

for $p \in (1,\infty)$, where we used (10) in the first and (6) in the second inequality, and $\|h_N\|_1 = 1$, which follows directly from (4), as well as $\hat{h}_N(0) = 1$. Further, h_N is continuous, because $h_N \in \mathcal{B}_{2\pi}^1$.

Let $\mathcal{A} \subset \mathbb{N}$ be a recursively enumerable nonrecursive set and $\phi_{\mathcal{A}}\colon \mathbb{N} \to \mathcal{A}$ a recursive enumeration of the elements of $\mathcal{A}$, where $\phi_{\mathcal{A}}$ is a one-to-one function. Further, let

$$f_*(t) = \sum_{N=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} h_N(t), \quad t \in \mathbb{R}. \tag{12}$$

Since

$$\sum_{N=1}^{\infty} \left\| \frac{1}{2^{\phi_{\mathcal{A}}(N)}} h_N \right\|_{\mathcal{B}_{2\pi}^1} = \sum_{N=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} < \sum_{N=1}^{\infty} \frac{1}{2^N} = 1,$$

we see that the series in (12) converges in the $\mathcal{B}_{2\pi}^1$ norm, and that $f_* \in \mathcal{B}_{2\pi}^1$. Note that $f_* \in \mathcal{B}_{2\pi}^1$ implies that $\hat{f}_*$ is continuous. Moreover, for $p \in (1,\infty)$ and $M \in \mathbb{N}$, we have

$$\begin{aligned}\left\| f_* - \sum_{N=1}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} h_N \right\|_{\mathcal{B}_{2\pi}^p} &\leq \sum_{N=M+1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} \|h_N\|_{\mathcal{B}_{2\pi}^p} \\ &< \sum_{N=M+1}^{\infty} \frac{C_1(p)}{2^N \log(N+1)} \\ &< \frac{C_1(p)}{\log(M+2)} \sum_{N=M+1}^{\infty} \frac{1}{2^N} \\ &< \frac{C_1(p)}{\log(M+2)}, \end{aligned}$$

where we used (11). This shows that, for $p \in (1,\infty)\cap\mathbb{R}_c$, the computable sequence

$$\left\{ \sum_{N=1}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} h_N \right\}_{M=1}^{\infty}$$

converges effectively in the $\mathcal{B}_{2\pi}^p$ norm to f_*. Hence, f_* is computable in $\mathcal{B}_{2\pi}^p$ for all $p \in (1,\infty)\cap\mathbb{R}_c$. f_* is also computable in $\mathcal{B}_{2\pi,0}^{\infty}$ because we have $\|f\|_{\mathcal{B}_{2\pi,0}^{\infty}} \leq (1+2\pi)\|f\|_{\mathcal{B}_{2\pi}^p}$ for all $f \in \mathcal{B}_{2\pi}^p$, according to Nikol'skiĭ's inequality [17, p. 49].

Since $f_* \in \mathcal{B}_{2\pi}^1$ and $h_N(t) \geq 0$ for all $t \in \mathbb{R}$, it follows from Lebesgue's dominated convergence theorem that

$$\hat{f}_*(\omega) = \sum_{N=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} \hat{h}_N(\omega).$$

Hence, we see that

$$\hat{f}_*(0) = \sum_{N=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}},$$

which implies that $\hat{f}_*(0) \notin \mathbb{C}_c$ according to Lemma 1.

Let $\omega \in (-2\pi, 2\pi)\setminus\{0\}$ be arbitrary but fixed and $\delta = \min\{1/2, |\omega|, 2\pi-\omega, 2\pi+\omega\}$. Then we have

$$\begin{aligned}|\hat{h}_N(\omega)| &= \frac{\hat{q}(\omega)}{\hat{g}_N(0)} \left| \sum_{k=1}^{N} \frac{1}{k}\cos(k\omega) - i\sum_{k=1}^{N}\frac{1}{k}\sin(k\omega) \right| \\ &< \frac{1}{\hat{g}_N(0)} \underbrace{\left(\log\left(\frac{1}{\delta}\right) + 2 + 3\pi \right)}_{=:C_2(\delta)} \\ &\leq \frac{C_2(\delta)}{\log(N+1)}, \end{aligned}$$

where we used Lemmas 2 and 3 in the first inequality. It follows that

$$\begin{aligned}\left| \hat{f}_*(\omega) - \sum_{N=1}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} \hat{h}_N(\omega) \right| &\leq \sum_{N=M+1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} |\hat{h}_N(\omega)| \\ &< \sum_{N=M+1}^{\infty} \frac{1}{2^N} |\hat{h}_N(\omega)| \\ &\leq C_2(\delta) \sum_{N=M+1}^{\infty} \frac{1}{2^N \log(N+1)} \end{aligned}$$

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$$< \frac{C_2(\delta)}{\log(M+2)} \sum_{N=M+1}^{\infty} \frac{1}{2^N}$$
$$< \frac{C_2(\delta)}{\log(M+2)}.$$

For $\omega \in (-2\pi, 2\pi) \cap \mathbb{R}_c \setminus \{0\}$, the constant $C_2(\delta)$ is computable, and we see that the sequence

$$\left\{ \sum_{N=1}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} \hat{h}_N(\omega) \right\}_{M=1}^{\infty}$$

of computable numbers converges effectively to $\hat{f}_*(\omega)$. This shows that $\hat{f}_*(\omega)$ is computable for all $\omega \in (-2\pi, 2\pi) \cap \mathbb{R}_c \setminus \{0\}$. Since $f_* \in \mathcal{B}^1_{2\pi}$ and f_* is continuous, we have $\hat{f}_*(\omega) = 0$ for all $|\omega| \geq 2\pi$. Hence, it follows that $\hat{f}_*(\omega)$ is computable for all $\omega \in \mathbb{R}_c \setminus \{0\}$. □

Remark 4. For $\omega \neq 0$ we have

$$|\hat{f}_*(\omega)| \leq \sum_{N=1}^{\infty} \frac{|\hat{g}_N(\omega)|}{2^{\phi_{\mathcal{A}}(N)} \hat{g}_N(0)} \leq \hat{q}(\omega) \sum_{N=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}}$$
$$\leq \sum_{N=1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} = \hat{f}_*(0),$$

which shows that $\hat{f}_*(0)$ is the maximum of the function $|\hat{f}_*|$. Since $\hat{f}_*(0)$ is not computable, it follows that $\|\hat{f}_*\|_C = \max_{\omega \in \mathbb{R}} |\hat{f}_*(\omega)|$ is not computable. This is interesting because we have $\hat{f}_*(\omega) \in \mathbb{C}_c$ for all $\omega \in \mathbb{R}_c \setminus \{0\}$ and $\lim_{\omega \to 0} \hat{f}_*(\omega) = \hat{f}_*(0)$.

VI. Algorithmic Decision

As we have seen in Theorem 1, there exist functions f in $\mathcal{B}^p_{2\pi}$, $1 < p < \infty$, $p \in \mathbb{R}_c$, that are computable in $\mathcal{B}^p_{2\pi}$, but their Fourier transform $\hat{f}$ is not computable in C. From a practical point of view it would be desirable to have an algorithm with which we can decide in advance for each function $f \in \mathcal{CB}^p_{2\pi}$ whether $\hat{f}$ is computable or not. This information would also be necessary for the application of computer aided control system design where we need to avoid problematic functions. In the following, we study the question if such an algorithm can be developed.

For $1 < p < \infty$, $p \in \mathbb{R}_c$, let

$$\mathcal{U}_p = \left\{ f \in \mathcal{B}^1_{2\pi} \cap \mathcal{CB}^p_{2\pi} : \hat{f} \text{ is computable} \right\}.$$

Thus, $\mathcal{U}_p$ is the set of benign functions, i.e., functions for which the Fourier transform is computable. Note that every function $f \in \mathcal{U}_p$ has a continuous Fourier transform $\hat{f}$, because $f \in \mathcal{B}^1_{2\pi}$. Now the question is: Does there exist a Turing machine $TM\colon \mathcal{B}^1_{2\pi} \cap \mathcal{CB}^p_{2\pi} \to \{0,1\}$ with $TM(f) = 1$ if and only if $f \in \mathcal{U}_p$? Such a machine would give us the answer to the question whether the Fourier transform of a function $f \in \mathcal{B}^1_{2\pi} \cap \mathcal{CB}^p_{2\pi}$ is computable or not. If the output of the machine is "1" then the Fourier transform is computable, if the output is "0" then the Fourier transform is not computable.

The next theorem answers this question about the existence of such a Turing machine in the negative. Note that since $\mathcal{B}^1_{2\pi} \subset \mathcal{B}^p_{2\pi}$, $1 < p < \infty$, the set $\mathcal{B}^1_{2\pi} \cap \mathcal{CB}^p_{2\pi}$ is relatively small. Yet we cannot always algorithmically decide if a function in this set has a computable Fourier transform or not.

Theorem 2. *Let $1 < p < \infty$, $p \in \mathbb{R}_c$. There exists no Turing machine that can decide for all $f \in \mathcal{B}^1_{2\pi} \cap \mathcal{CB}^p_{2\pi}$ whether $f \in \mathcal{U}_p$ or $f \notin \mathcal{U}_p$.*

Proof. Let $1 < p < \infty$, $p \in \mathbb{R}_c$, and f_* be the function from Theorem 1 and $g \in \mathcal{U}_p$. We have $f_* \in \mathcal{B}^1_{2\pi} \cap \mathcal{CB}^p_{2\pi} \setminus \mathcal{U}_p$. For $\mu \in [-1,1] \cap \mathbb{R}_c$, we consider the function $F_\mu = \mu f_* + (1-\mu) g$ and set $\psi(\mu) = \chi_{\mathcal{U}_p}(F_\mu)$, where $\chi_{\mathcal{U}_p}$ denotes the characteristic function of the set $\mathcal{U}_p$. For $\mu = 0$ we have $F_0 = g \in \mathcal{U}_p$. For all other μ, i.e., $\mu \in [-1,1] \cap \mathbb{R}_c \setminus \{0\}$, we have $F_\mu \in \mathcal{B}^1_{2\pi} \cap \mathcal{CB}^p_{2\pi} \setminus \mathcal{U}_p$, because $f_* \in \mathcal{B}^1_{2\pi} \cap \mathcal{CB}^p_{2\pi} \setminus \mathcal{U}_p$. Thus, we see that

$$\psi(\mu) = \begin{cases} 1, & \mu = 0, \\ 0, & \mu \in [-1,1] \cap \mathbb{R}_c \setminus \{0\}. \end{cases}$$

It follows that for every computable sequence $\{\mu_n\}_{n \in \mathbb{Z}}$ of real numbers with $\lim_{n \to \infty} \mu_n = 0$ and $\mu_n \neq 0$, $n \in \mathbb{N}$, we have

$$0 = \lim_{n \to \infty} \psi(\mu_n) < \psi(0) = 1,$$

i.e., ψ is a discontinuous function on $[-1,1] \cap \mathbb{R}_c$. This implies that ψ is not Banach–Mazur computable, because every Banach–Mazur computable function is necessarily continuous [19].

We prove the assertion by contradiction. Assume that there exists a Turing machine that for all $f \in \mathcal{B}^1_{2\pi} \cap \mathcal{CB}^p_{2\pi}$ can decide whether $f \in \mathcal{U}_p$. This means we can construct a Turing machine $TM_1\colon \mathcal{B}^1_{2\pi} \cap \mathcal{CB}^p_{2\pi} \to \{0,1\}$ with $TM_1(f) = 1$ if and only if $f \in \mathcal{U}_p$.

For $\mu \in [-1,1] \cap \mathbb{R}_c$, we have $\psi(\mu) = \chi_{\mathcal{U}_p}(F_\mu) = TM(F_\mu)$. Further, since F_μ is computable, there exists a Turing machine $TM_2\colon [-1,1] \cap \mathbb{R}_c \to \mathcal{B}^1_{2\pi} \cap \mathcal{CB}^p_{2\pi}$ with $TM_2(\mu) = F_\mu$. It follows that the concatenation of both Turing machines gives a Turing machine $TM_3\colon [-1,1] \cap \mathbb{R}_c \to \{0,1\}$ with $TM_3(\mu) = TM_1(TM_2(\mu)) = TM_1(F_\mu) = \psi(\mu)$. Let $\{\lambda_n\}_{n \in \mathbb{N}}$ be a computable sequence of real numbers. Then $\{q_n\}_{n \in \mathbb{N}}$ with $q_n = TM_3(\lambda_n) = \psi(\lambda_n)$, $n \in \mathbb{N}$, is a computable sequence. Hence, ψ maps the computable sequence $\{\lambda_n\}_{n \in \mathbb{N}}$ into the computable sequence $\{\psi(\lambda_n)\}_{n \in \mathbb{N}}$, or, in other words, ψ is Banach–Mazur computable. This is a contradiction, because above we have already shown that ψ is not Banach–Mazur computable. □

VII. Discrete-Time Fourier Transform

The discrete-time Fourier transform (DTFT) of a sequence $x = \{x(k)\}_{k \in \mathbb{Z}}$ is defined as

$$X(\mathrm{e}^{i\omega}) = \sum_{k=-\infty}^{\infty} x(k)\, \mathrm{e}^{-i\omega k}. \tag{13}$$

The DTFT enables us to analyze sequences in the frequency domain, and therefore is an essential tool in signal processing [5], [34], [35]. A practical and very important fact is the convolution theorem of the DTFT. Let

$$(x * y)(l) = \sum_{k=-\infty}^{\infty} x(l-k) y(k)$$

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denote the convolution of the two sequences x and y. Since $DTFT(x*y)(\omega) = X(\mathrm{e}^{i\omega})Y(\mathrm{e}^{i\omega})$, the convolution can be calculated in the frequency domain, according to the convolution theorem of the DTFT

$$(x*y)(l) = DTFT^{-1}[DTFT[x]DTFT[y]](l). \qquad (14)$$

In many cases this allows a more efficient implementation of the convolution.

A very closely related transform is the discrete Fourier transform (DFT), which is the basis of many applications in signal processing and modern communications [2], [3], [5], [36]. For the DFT only finite segments of the discrete-time signal are considered. Then the infinite sum in (13) reduces to a finite sum, and X has to be evaluated only for a finite set of discrete frequencies. Thus, the DFT is well suited for being implemented on digital computers. In particular, its implementation in the form of the FFT algorithm is widely used [37]–[39].

A crucial point in any implementation of the DTFT is that X can be effectively approximated, for example by a finite series. This means, for any given prescribed error ϵ, we need to be able to approximate X by an algorithm in a computable number of steps, such that the approximation error is guaranteed to be less than ϵ. This kind of error control is only possible if X is computable.

From a signal theoretic point of view, we are used to "equivalence" between time and frequency domain, and theorems like the convolution theorem enable us to make efficient use of the frequency domain representation of discrete signals. In the next section we will analyze whether this "equivalence" between time domain and frequency domain still holds from a computational perspective. This is relevant if we want to implement an algorithm, such as (14), on a digital computer.

As outlined in the introduction, we are interested in whether the DTFT of a well-behaved sequence is always computable. In this paper we will only consider sequences in $\ell^p(\mathbb{Z})$, $1 \leq p < \infty$. The weakest requirement, i.e., a necessary condition for computability, is that $X(\mathrm{e}^{i\omega}) \in \mathbb{C}_c$ for all $\omega \in [0, 2\pi) \cap \mathbb{R}_c$. By constructing a counter example, we show that this is not always the case.

Theorem 3. *We can construct a sequence x_* with the following properties:*

1) *$x_* \in \ell^p_+(\mathbb{Z})$ for all $1 \leq p < \infty$,*
2) *$x_* \in \mathcal{C}\ell^p_+(\mathbb{Z})$, i.e, x_* is computable as an element of $\ell^p_+(\mathbb{Z})$ for all $1 < p < \infty$, $p \in \mathbb{R}_c$,*
3) *$X_* \in \mathcal{W}$,*
4) *$X_*(\mathrm{e}^{i\omega})$ is absolutely continuous, i.e., the derivative $\frac{\mathrm{d}}{\mathrm{d}\omega}X_*(\mathrm{e}^{i\omega})$ exists almost everywhere with respect to the Lebesgue measure, we have $\frac{\mathrm{d}}{\mathrm{d}\omega}X_*(\mathrm{e}^{i\omega}) \in L^1(\partial\mathbb{D})$ and X_* can be represented as the integral of $\frac{\mathrm{d}}{\mathrm{d}\omega}X_*(\mathrm{e}^{i\omega})$,*
5) *$X_*(1) \notin \mathbb{C}_c$ and $X_*(\mathrm{e}^{i\omega}) \in \mathbb{C}_c$ for all $\omega \in (0, 2\pi) \cap \mathbb{R}_c$,*
6) *$\|X_*\|_C$ is not computable.*

According to Theorem 3, for the computable sequence x_* we cannot compute the DTFT X_* as a function, because $X_*(1)$ is not computable. This implies that we cannot algorithmically compute X_* on a digital computer with control of the approximation error.

We can also interpret Theorem 3 as a representation result for continuous 2π-periodic functions. In this light, $X(\mathrm{e}^{-i\omega})$ is a continuous 2π-periodic function and x are its Fourier coefficients. Theorem 3 shows that although the sequence of Fourier coefficients x is computable and has very nice properties, the corresponding continuous function $X(\mathrm{e}^{-i\omega})$, defined by the Fourier series, is not computable because $X(1) \notin \mathbb{C}_c$.

If we do not require the properties 3, 4, and 6 to hold, we can even strengthen the non-computability statement of the theorem.

Theorem 4. *We can construct a sequence x_{**} with the following properties:*

1) *$x_{**} \in \ell^p_+(\mathbb{Z})$ for all $1 \leq p < \infty$,*
2) *$x_{**} \in \mathcal{C}\ell^p_+(\mathbb{Z})$, i.e., x_{**} is computable as an element of $\ell^p_+(\mathbb{Z})$, for all $1 < p < \infty$, $p \in \mathbb{R}_c$,*
3) *For all $M \in \mathbb{N}$ we have for all $0 \leq l \leq 2^M - 1$ that $X_{**}(\mathrm{e}^{il2\pi/2^M}) \notin \mathbb{C}_c$, i.e., $X_{**}(\mathrm{e}^{i\omega})$ is not computable for all frequencies ω on all dyadic grids.*

We call any subset of $[0, 2\pi]$ that, for some $M \in \mathbb{N}$, has the form $\{l2\pi/2^M : 0 \leq l \leq 2^M - 1\}$, a dyadic grid. Theorem 4 implies that the inverse DTFT integral

$$x_{**}(k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X_{**}(\mathrm{e}^{i\omega})\,\mathrm{e}^{i\omega k}\,\mathrm{d}\omega, \quad k \in \mathbb{Z},$$

cannot be evaluated numerically as a Riemann sum on the dyadic grid, because X_{**} is not computable at these points. In general, since X_{**} is not computable on the dyadic grid, all approaches that rely on these numbers, like the inverse DFT [4], [40], [41], cannot be used.

Since x_{**} is computable in $\ell^2_+(\mathbb{Z})$, it follows that X_{**} is computable in $L^2(\partial\mathbb{D})$, i.e., X_{**} can be effectively approximated by computable trigonometric polynomials. However, these polynomials have to be constructed first, and the values of X_{**} on the dyadic grid, i.e., the values $X_{**}(\mathrm{e}^{il2\pi/2^M})$, $0 \leq l \leq 2^M - 1$, cannot be used for this construction, because they are not computable.

For the proof of Theorem 3, we need two lemmas.

Lemma 4. *For all $\omega \in (0, 2\pi)$, there exists a constant $C_3(\omega) < \infty$, such that*

$$\sum_{k=2}^{N} \frac{1}{k\log(k)}\cos(k\omega) \leq C_3(\omega)$$

for all $N \in \mathbb{N}$. For $\omega \in (0, 2\pi) \cap \mathbb{R}_c$, we have $C_3(\omega) \in \mathbb{R}_c$.

Lemma 5. *There exists a continuous 2π-periodic function $Q(\omega)$, such that*

$$\lim_{N\to\infty} \max_{\omega\in[0,2\pi)} \left| Q(\omega) - \sum_{k=2}^{N} \frac{1}{k\log(k)}\sin(k\omega) \right| = 0.$$

Further, there exists a computable constant $C_4 \in \mathbb{R}_c$ such that

$$\left| \sum_{k=2}^{N} \frac{1}{k\log(k)}\sin(k\omega) \right| \leq C_4$$

for all $\omega \in [0, 2\pi)$.

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Lemmas 4 and 5 are known results about trigonometric series, and can, for example, be found in [13].

Now we are in the position to prove Theorem 3.

Proof of Theorem 3. Let $N \in \mathbb{N}$, $N \geq 2$ be arbitrary. We consider

$$x_N(k) = \begin{cases} \frac{C(N)}{k\log(k)}, & 2 \leq k \leq N, \\ 0, & \text{otherwise}, \end{cases}$$

where

$$C(N) = \left(\sum_{k=2}^{N} \frac{1}{k\log(k)}\right)^{-1}.$$

For all $k \in \mathbb{N}$, $k\log(k)$ is a computable number. Therefore, $C(N)$ is computable. Hence, x_N is a computable sequence of computable numbers. Since

$$\frac{1}{k\log(k)} > \int_k^{k+1} \frac{1}{\tau\log(\tau)}\,\mathrm{d}\tau$$

for all $k \geq 2$, we have

$$\sum_{k=2}^{N} \frac{1}{k\log(k)} > \int_2^{N+1} \frac{1}{\tau\log(\tau)}\,\mathrm{d}\tau = \log\left(\frac{\log(N+1)}{\log(2)}\right),$$

and it follows that

$$C(N) < \left(\log\left(\frac{\log(N+1)}{\log(2)}\right)\right)^{-1} < 3. \tag{15}$$

Further, for $p \in [1,\infty) \cap \mathbb{R}_c$, we have $x_N \in \mathcal{C}\ell^p_+(\mathbb{Z})$ and $\|x_N\|_{\ell^p}$ is computable. For $p > 1$, we have

$$\begin{aligned} \|x_N\|^p_{\ell^p} &< (C(N))^p \sum_{k=1}^{N} \frac{1}{k^p} \\ &< (C(N))^p \left(1 + \sum_{k=2}^{N} \int_{k-1}^{k} \frac{1}{\tau^p}\,\mathrm{d}\tau\right) \\ &< (C(N))^p \left(1 + \frac{1}{p-1}\right), \end{aligned} \tag{16}$$

where we used

$$1/k^p < \int_{k-1}^{k} 1/\tau^p\,\mathrm{d}\tau$$

for all $k \geq 2$ in the second inequality. For $p = 1$, we have

$$\|x_N\|_{\ell^1} = 1. \tag{17}$$

The DTFT of the sequence x_N is given by

$$X_N(\mathrm{e}^{i\omega}) = \sum_{k=2}^{N} x_N(k)\,\mathrm{e}^{-ik\omega}, \quad \omega \in [0,2\pi),$$

and we have

$$|X_N(\mathrm{e}^{i\omega})| \leq \sum_{k=2}^{N} |x_N(k)| = 1 \tag{18}$$

for all $\omega \in [0,2\pi)$, as well as

$$X_N(1) = \sum_{k=2}^{N} x_N(k) = 1.$$

Hence, it follows that

$$\max_{\omega\in[0,2\pi)} |X_N(\mathrm{e}^{i\omega})| = X_N(1) = 1.$$

Let

$$A_N(\omega) = C(N) \sum_{k=2}^{N} \frac{1}{k\log(k)} \cos(k\omega) \tag{19}$$

and

$$B_N(\omega) = C(N) \sum_{k=2}^{N} \frac{1}{k\log(k)} \sin(k\omega). \tag{20}$$

Then we have

$$X_N(\mathrm{e}^{i\omega}) = A_N(\omega) - iB_N(\omega).$$

Let $\mathcal{A} \subset \mathbb{N}$ be an arbitrary recursively enumerable nonrecursive set [20]. Further, let $\phi_{\mathcal{A}} \colon \mathbb{N} \to \mathcal{A}$ be a recursive enumeration of the set $\mathcal{A}$, such that for every element $k \in \mathcal{A}$ there exists exactly one $N_k \in \mathbb{N}$ with $\phi_{\mathcal{A}}(N_k) = k$. We consider

$$X_*(\mathrm{e}^{i\omega}) = \sum_{N=2}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} X_N(\mathrm{e}^{i\omega}), \quad \omega \in [0,2\pi), \tag{21}$$

and

$$x_*(k) = \sum_{N=2}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} x_N(k), \quad k \in \mathbb{Z}. \tag{22}$$

For $p > 1$, using (16) and (15), we obtain

$$\begin{aligned} \|x_*\|_{\ell^p} &\leq \sum_{N=2}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} \|x_N\|_{\ell^p} \\ &\leq \left(1 + \frac{1}{p-1}\right)^{\frac{1}{p}} \sum_{N=2}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} C(N) \\ &< 3\left(1 + \frac{1}{p-1}\right)^{\frac{1}{p}} \sum_{N=2}^{\infty} \frac{1}{2^N} \\ &= \frac{3}{2}\left(1 + \frac{1}{p-1}\right)^{\frac{1}{p}} \end{aligned}$$

and, for $p = 1$, using (17),

$$\|x_*\|_{\ell^1} \leq \sum_{N=2}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} \|x_N\|_{\ell^1} < \frac{1}{2}.$$

This proves item 1, as well as item 3.

Further, for $p \in (1,\infty) \cap \mathbb{R}_c$ and $M \geq 2$, we have

$$\begin{aligned} \left\| x_* - \sum_{N=2}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} x_N \right\|_{\ell^p} &= \left\| \sum_{N=M+1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} x_N \right\|_{\ell^p} \\ &\leq \left(1 + \frac{1}{p-1}\right)^{\frac{1}{p}} \sum_{N=M+1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} C(N) \\ &< C(M+1)\left(1 + \frac{1}{p-1}\right)^{\frac{1}{p}} \sum_{N=M+1}^{\infty} \frac{1}{2^N} \\ &< \frac{1}{2}\left(\log\left(\frac{\log(M+1)}{\log(2)}\right)\right)^{-1} \left(1 + \frac{1}{p-1}\right)^{\frac{1}{p}}, \end{aligned}$$

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where we used $C(N) > C(N+1)$ for all $N \geq 2$. Thus, we see that the computable sequence

$$\left\{\sum_{N=2}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} x_N\right\}_{M=2}^{\infty}$$

converges effectively to x_*. Hence, x_* is computable in $\ell_+^p(\mathbb{Z})$, $p \in (1,\infty) \cap \mathbb{R}_c$. This proves item 2.

Further, the sequence in (21) convergences absolutely and uniformly, and thus we have $X_* \in C(\partial\mathbb{D})$. For $\omega \in (0, 2\pi)$, we have

$$\begin{aligned}|X_*(\mathrm{e}^{i\omega})| &\leq \sum_{N=2}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} |X_N(\mathrm{e}^{i\omega})| \\ &\leq \sum_{N=2}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} \\ &= X_*(1),\end{aligned}$$

where we used (18) in the second inequality. It follows that

$$\|X_*\|_C = \max_{[0,2\pi)} |X_*(\mathrm{e}^{i\omega})| = X_*(1).$$

Since we have

$$\sum_{N=2}^{\infty} 2^{-\phi_{\mathcal{A}}(N)} \notin \mathbb{R}_c,$$

it follows that $X_*(1) \neq \mathbb{C}_c$ and $\|X_*\|_C \neq \mathbb{C}_c$. This proves item 6 and the first part of item 5. Let

$$A(\omega) = \sum_{N=2}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} A_N(\omega) \tag{23}$$

and

$$B(\omega) = \sum_{N=2}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} B_N(\omega). \tag{24}$$

The series in (23) and (24) are absolutely convergent. Further, for $\omega \in [0, 2\pi)$ and $M \geq 2$, we have

$$\begin{aligned}\left|B(\omega) - \sum_{N=2}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} B_N(\omega)\right| &= \left|\sum_{N=M+1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} B_N(\omega)\right| \\ &\leq \sum_{N=M+1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} |B_N(\omega)| \\ &\leq C_4 \sum_{N=M+1}^{\infty} \frac{C(N)}{2^{\phi_{\mathcal{A}}(N)}} \\ &< C_4 \left(\log\left(\frac{\log(M+1)}{\log(2)}\right)\right)^{-1} \sum_{N=M+1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} \\ &< \frac{C_4}{2} \left(\log\left(\frac{\log(M+1)}{\log(2)}\right)\right)^{-1},\end{aligned}$$

where we used Lemma 5 in the second inequality. Note that C_4 is a computable number. Thus, the computable sequence

$$\left\{\sum_{N=2}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} B_N(\omega)\right\}_{M=2}^{\infty}$$

of computable numbers converges effectively to $B(\omega)$. Hence, we have $B(\omega) \in \mathbb{R}_c$ for all $\omega \in [0, 2\pi) \cap \mathbb{R}_c$. For $\omega \in (0, 2\pi)$ and $M \geq 2$, we have

$$\begin{aligned}\left|A(\omega) - \sum_{N=2}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} A_N(\omega)\right| &= \left|\sum_{N=M+1}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} A_N(\omega)\right| \\ &< \frac{C_3(\omega)}{2\log\left(\frac{\log(M+1)}{\log(2)}\right)},\end{aligned} \tag{25}$$

where we used Lemma 4 in the second line. For $\omega \in (0, 2\pi) \cap \mathbb{R}_c$ we have $C_3(\omega) \in \mathbb{R}_c$. Thus, for $\omega \in (0, 2\pi) \cap \mathbb{R}_c$, the computable sequence

$$\left\{\sum_{N=2}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} A_N(\omega)\right\}_{M=2}^{\infty}$$

of computable numbers converges effectively to $A(\omega)$. Hence, we have $A(\omega) \in \mathbb{R}_c$ for all $\omega \in (0, 2\pi) \cap \mathbb{R}_c$. Thus, we see that

$$X_*(\mathrm{e}^{i\omega}) = A(\omega) - iB(\omega)$$

is computable for all $\omega \in (0, 2\pi) \cap \mathbb{R}_c$. This proves the second part of item 5.

Last, we prove item 4. Let

$$K_N(\omega) = X_N(\mathrm{e}^{i\omega}), \quad \omega \in [0, 2\pi).$$

Since K_N is absolutely continuous, the derivative K_N' exists almost everywhere with respect to the Lebesgue measure, and we have $K_N' \in L^1([0, 2\pi))$ as well as

$$K_N(\omega) = \int_0^{\omega} K_N'(\xi)\,\mathrm{d}\xi + K_N(0).$$

According to the definition of X_* in (21), we have

$$X_*(\mathrm{e}^{i\omega}) = \sum_{N=2}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} K_N(\omega).$$

Let

$$G(\omega) = \sum_{N=2}^{\infty} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} K_N'(\omega). \tag{26}$$

Since

$$\|K_N'\|_{L^1([0,2\pi))} = \|A_N' - iB_N'\|_{L^1([0,2\pi))},$$

it follows from Lemma 6, which we state and prove later, that there exists a constant C_5 such that

$$\|K_N'\|_{L^1([0,2\pi))} \leq C_5 \tag{27}$$

for all N. It follows that the series in (26) converges in the $L^1([0, 2\pi))$-norm and that $G \in L^1([0, 2\pi))$. Further, we have

$$\begin{aligned}&\sum_{N=2}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} K_N(\omega) \\ &\quad= \sum_{N=2}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} \left(\int_0^{\omega} K_N'(\xi)\,\mathrm{d}\xi + K_N(0)\right) \\ &\quad= \int_0^{\omega} \sum_{N=2}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} K_N'(\xi)\,\mathrm{d}\xi + \sum_{N=2}^{M} \frac{1}{2^{\phi_{\mathcal{A}}(N)}} K_N(0).\end{aligned} \tag{28}$$

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Let

$$U(\omega)=\int_0^\omega G(\xi)\,\mathrm{d}\xi+\sum_{N=2}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(N)}}K_N(0). \tag{29}$$

Then U is absolutely continuous, according to the fundamental theorem of Lebesgue integral calculus. From (28) and (29) we see that

$$\begin{aligned}&\left|U(\omega)-\sum_{N=2}^{M}\frac{1}{2^{\phi_{\mathcal{A}}(N)}}K_N(\omega)\right|\\&=\left|\int_0^\omega\left(\sum_{N=2}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(N)}}K_N'(\xi)-\sum_{N=2}^{M}\frac{1}{2^{\phi_{\mathcal{A}}(N)}}K_N'(\xi)\right)\mathrm{d}\xi\right|\\&=\left|\sum_{N=M+1}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(N)}}\int_0^\omega K_N'(\xi)\,\mathrm{d}\xi\right|\\&\leq\sum_{N=M+1}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(N)}}\int_0^\omega|K_N'(\xi)|\,\mathrm{d}\xi\\&\leq C_5\sum_{N=M+1}^{\infty}\frac{1}{2^N},\end{aligned}$$

where we used (27) in the last inequality. This shows that

$$U(\omega)=\lim_{M\to\infty}\sum_{N=2}^{M}\frac{1}{2^{\phi_{\mathcal{A}}(N)}}K_N(\omega)=X_*(\mathrm{e}^{i\omega})$$

for all $\omega\in[0,2\pi)$. Hence, $X_*(\mathrm{e}^{i\omega})$ is absolutely continuous. □

Remark 5. The sequence $\{X_N\}_{N\in\mathbb{N}}$ in the proof of Theorem 3 converges uniformly to X_*, but the approximation error cannot be effectively controlled, i.e., to a given prescribed error it is not possible to algorithmically compute an index M such that X_M achieves this error.

We state and prove Lemma 6 next.

Lemma 6. *There exist two constants C_6 and C_7 such that*

$$\|A_N'\|_{L^1([0,2\pi))}\leq C_6$$

and

$$\|B_N'\|_{L^1([0,2\pi))}\leq C_7$$

for all $N\in\mathbb{N}$.

Proof. We have

$$A_N'(\omega)=-C(N)\sum_{k=2}^{N}\frac{1}{\log(k)}\sin(k\omega)$$

and

$$B_N'(\omega)=C(N)\sum_{k=2}^{N}\frac{1}{\log(k)}\cos(k\omega).$$

Using the bounds from [32, pp. 182–192], it can be shown that

$$\left\|\sum_{k=2}^{N}\frac{1}{\log(k)}\sin(k\,\cdot)\right\|_1\sim C_8\log(\log(N))$$

and that there exists a constant C_9 such that

$$\left\|\sum_{k=2}^{N}\frac{1}{\log(k)}\cos(k\,\cdot)\right\|_1\leq C_9$$

for all $N\in\mathbb{N}$, $N\geq 2$. Since

$$C(N)<\left(\log\left(\frac{\log(N+1)}{\log(2)}\right)\right)^{-1},$$

the assertion follows. □

Finally, we prove Theorem 4.

Proof of Theorem 4. For $K\in\mathbb{N}$, let

$$x_K(k)=\begin{cases}\frac{1}{K+1}, & \text{if } k=2^m \text{ for some } m\in[K,2K],\\ 0, & \text{otherwise.}\end{cases}$$

Then we have

$$X_K(\mathrm{e}^{i\omega})=\sum_{k=0}^{\infty}x_K(k)\,\mathrm{e}^{-in\omega}=\frac{1}{K+1}\sum_{k=K}^{2K}\mathrm{e}^{-i\omega 2^k}.$$

For $M\in\mathbb{N}$, $K\geq M$ we have for $0\leq l\leq 2^M-1$ that

$$X_K(\mathrm{e}^{i\frac{l2\pi}{2^M}})=\frac{1}{K+1}\sum_{k=K}^{2K}\mathrm{e}^{-i\frac{l2\pi}{2^M}2^k}=\frac{1}{K+1}\sum_{k=K}^{2K}\mathrm{e}^{i0}=1.$$

Let $\mathcal{A}\subset\mathbb{N}$ be a recursively enumerable nonrecursive set and $\phi_{\mathcal{A}}\colon\mathbb{N}\to\mathcal{A}$ a recursive enumeration of the elements of $\mathcal{A}$, where $\phi_{\mathcal{A}}$ is a one-to-one function. We consider

$$x_{**}(k)=\sum_{K=1}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(K)}}x_K(k),\quad k\in\mathbb{Z}. \tag{30}$$

The series in (30) converges in $\ell^1_+(\mathbb{Z})$, and, using similar calculations as in the proof of Theorem 3, it is shown that x_{**} is computable in $\mathcal{C}\ell^p_+(\mathbb{Z})$ for $1<p<\infty$, $p\in\mathbb{R}_c$. Further, we have

$$X_{**}(\mathrm{e}^{i\omega})=\sum_{n=0}^{\infty}x_{**}(n)\,\mathrm{e}^{-in\omega}=\sum_{K=1}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(K)}}X_K(\mathrm{e}^{i\omega}),$$

where both series are absolutely convergent. Let $M\in\mathbb{N}$ be arbitrary, and consider $0\leq l\leq 2^M-1$. We have

$$\begin{aligned}X_{**}(\mathrm{e}^{i\frac{l2\pi}{2^M}})&=\sum_{K=1}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(K)}}X_K(\mathrm{e}^{i\frac{l2\pi}{2^M}})\\&=\sum_{K=1}^{M-1}\frac{1}{2^{\phi_{\mathcal{A}}(K)}}X_K(\mathrm{e}^{i\frac{l2\pi}{2^M}})+\sum_{K=M}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(K)}}\\&=C(l,M)+\sum_{K=M}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(K)}},\end{aligned}$$

because $X_K(\mathrm{e}^{i\frac{l2\pi}{2^M}})=1$ for $K\geq M$. $C(l,M)$ is a finite sum of computable complex numbers and hence computable. For every $M\in\mathbb{N}$ the number

$$\sum_{K=M}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(K)}}$$

is not computable, because

$$\sum_{K=M}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(K)}}=\sum_{K=1}^{\infty}\frac{1}{2^{\phi_{\mathcal{A}}(K)}}-\sum_{K=1}^{M-1}\frac{1}{2^{\phi_{\mathcal{A}}(K)}}, \tag{31}$$

and the first sum on the right hand side of (31) is not computable while the second sum is computable. □

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VIII. Application I

The output of an LTI system can either be calculated in the time domain as a convolution or in the frequency domain by multiplying the system input with the transfer function of the LTI system, as illustrated in Fig. 3. In this section we use the result from the previous section to show that there is no duality between the time domain and frequency domain with respect to computability.

We consider energetically stable LTI systems and the input signal space $\ell^2_+(\mathbb{Z})$. Let h_T denote the impulse response of the LTI system. We assume that $h_T \in \ell^2_+(\mathbb{Z})$, i.e., that the system is causal, and that $H_T \in L^\infty(\partial\mathbb{D})$. Even further, we restrict ourselves to such systems for which $H_T \in C(\partial\mathbb{D})$. Then the system output y is given by

$$y(k) = \sum_{l=0}^{k} h_T(k-l)x(l) = \sum_{l=0}^{k} h_T(l)x(k-l), \quad k \in \mathbb{N}.$$

If $h_T \in \ell^2_+(\mathbb{Z})$ and $x \in \ell^2_+(\mathbb{Z})$ are computable as sequences in $\ell^2_+(\mathbb{Z})$, then $y(k)$ is computable for every $k \in \mathbb{N}$, because both sums above are finite. Let $h_{T_*}(k) = x_*(k)$, $k \in \mathbb{N}$, where x_* is the sequence that was defined in (22). We have

$$|H_{T_*}(\mathrm{e}^{i\omega})| \leq \sum_{N=2}^{\infty} \frac{1}{2^{\phi_A(N)}} < \sum_{N=2}^{\infty} \frac{1}{2^N} = \frac{1}{2} < 1, \qquad (32)$$

for all $\omega \in [0, 2\pi)$. That is, the operator norm of the system T_* satisfies $\|T_*\| < 1$.

Next, we show that for all computable input sequences $x \in \ell^2_+(\mathbb{Z})$, the output sequence y is computable as an element of $\ell^2_+(\mathbb{Z})$. Let $x \in \ell^2_+(\mathbb{Z})$ be a computable sequence. Since $x \in \ell^2_+(\mathbb{Z})$ is computable, there exists a sequence $\{x_N\}_{N\in\mathbb{N}}$ with $x_N(k) \neq 0$ for only finitely many k and $x_N(k) \in \mathbb{C}_c$ for all $k \in \mathbb{N}$, as well as a recursive function $\xi\colon \mathbb{N} \to \mathbb{N}$, such that for all $m \in \mathbb{N}$ we have $\|x - x_N\|_{\ell^2} < 2^{-m}$ for all $N \geq \xi(m)$. Let $M_N = \max\{k \in \mathbb{N}\colon x_N(k) \neq 0\}$. Then we have

$$y_N(k) = \sum_{l=2}^{M_N} h_{T_*}(k-l)x_N(l).$$

Since $y_N(k)$ is computable for every $k \in \mathbb{Z}$, it follows that y_N, as a weighted sum of computable sequences, where the weights are computable, is a computable sequence. Using (32), we obtain

$$\begin{aligned}\|y - y_N\|^2_{\ell^2} &= \frac{1}{2\pi}\int_{-\pi}^{\pi} |H_{T_*}(\mathrm{e}^{i\omega})|^2 |X(\mathrm{e}^{i\omega}) - X_N(\mathrm{e}^{i\omega})|^2 \,\mathrm{d}\omega \\ &\leq \frac{1}{2\pi}\int_{-\pi}^{\pi} |X(\mathrm{e}^{i\omega}) - X_N(\mathrm{e}^{i\omega})|^2 \,\mathrm{d}\omega \\ &= \|x - x_N\|^2_{\ell^2}.\end{aligned}$$

Hence, for all $m \in \mathbb{N}$, we have

$$\|y - y_N\|_{\ell^2} \leq \|x - x_N\|_{\ell^2} < 2^{-m} \qquad (33)$$

for all $N \geq \xi(m)$. Although infinitely many elements of y_N are non-zero, it can be shown that (33) implies that y is computable in $\ell^2(\mathbb{Z})$.

We further have

$$Y(\mathrm{e}^{i\omega}) = H_{T_*}(\mathrm{e}^{i\omega})X(\mathrm{e}^{i\omega})$$

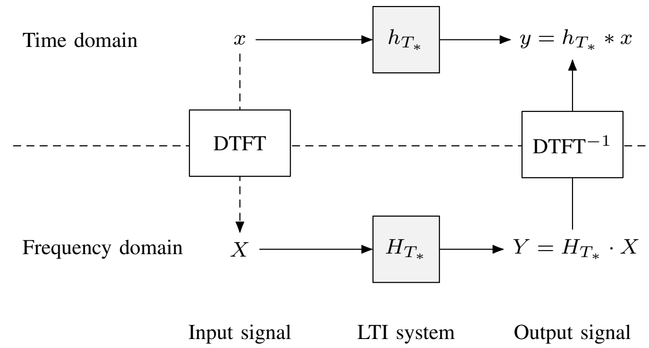

Fig. 3. Computation of the system output in the time and the frequency domain. With respect to computability there is no duality between the time and the frequency domain.

for almost all $\omega \in [0, 2\pi)$. However, we cannot use the DTFT to compute y, because H_{T_*} is not computable as an element of $C(\partial\mathbb{D})$. Thus, although the convolution is computable in the time domain according to above discussion, a computation of y via the frequency domain is not possible.

IX. Application II

The Poisson summation formula

$$\sum_{k=-\infty}^{\infty} f(k) = \sum_{k=-\infty}^{\infty} \hat{f}(k2\pi)$$

is frequently used, for example, to prove sampling theorems [42], [43]. By stating that the sum of the time domain samples equals the sum of the frequency domain samples, it connects the time and frequency domain. According to Poisson's summation formula, we have for functions $f \in \mathcal{B}^1_{2\pi}$, $1 \leq p < \infty$, that

$$\begin{aligned}\frac{1}{2}\sum_{k=-\infty}^{\infty} f\left(-\frac{k}{2}\right) \mathrm{e}^{i\omega k/2} &= \sum_{k=-\infty}^{\infty} \hat{f}(k4\pi + \omega) \\ &= \hat{f}(\omega), \quad \omega \in (-2\pi, 2\pi),\end{aligned}$$

where the last equality follows from the fact that $\hat{f}(\omega)$ is zero for $\omega \in \mathbb{R} \setminus [-2\pi, 2\pi]$. We know that $\hat{f}_*(\omega)$, is not computable as a function, because $\hat{f}_*(0)$ is not computable. Hence, it follows that

$$\frac{1}{2}\sum_{k=-\infty}^{\infty} f_*\left(-\frac{k}{2}\right) \mathrm{e}^{i\omega k/2}$$

is not computable either. This is surprising, since all components of this sum are computable.

X. Conclusion and Outlook

Since nowadays most computations are done on digital computers, the question of computability arises. In this paper, we analyzed the computability of the Fourier transform and the discrete-time Fourier transform, and proved that there exist well-behaved signals for which the transforms exist mathematically, but which are not Turing computable. Hence, the transforms of these signals cannot be computed on any digital hardware, such as CPUs, FPGAs, or DSPs. This result

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also implies that the usual duality between time and frequency domain does not hold with respect to computability. While the Fourier transform is not computable on a Turing machine, i.e., the theoretically ideal digital machine, it can be computed on an ideal analog machine, as discussed in Section IV. Whether and how this theoretical superiority of the analog machine can be translated into practice is unclear. Finding suitable analog implementations could be a goal of further investigations.

Appendix A
Derivation of the constant $C(p)$

Next, we will derive the explicit constant $C(p)$ in (7). We will sketch the main steps of the proof.

Let

$$(L_\pi g)(t) = \int_{-\infty}^{\infty} g(\tau)\frac{\sin(\pi(t-\tau))}{\pi(t-\tau)}\,\mathrm{d}\tau.$$

For $g \in L^p(\mathbb{R})$, $1 < p < \infty$, we have

$$\begin{aligned}(L_\pi g)(t) &= \int_{-\infty}^{\infty} g(\tau)\frac{\sin(\pi(t-k))}{\pi(t-k)}\,\mathrm{d}\tau \\ &= \frac{\mathrm{e}^{i\pi t}}{2\pi i}\text{V.P.}\int_{-\infty}^{\infty}\frac{g(\tau)\,\mathrm{e}^{-i\pi\tau}}{t-\tau}\,\mathrm{d}\tau \\ &\qquad - \frac{\mathrm{e}^{-i\pi t}}{2\pi i}\text{V.P.}\int_{-\infty}^{\infty}\frac{g(\tau)\,\mathrm{e}^{i\pi\tau}}{t-\tau}\,\mathrm{d}\tau \\ &= \frac{\mathrm{e}^{i\pi t}}{2i}H(g\,\mathrm{e}^{-i\pi\,\cdot})(t) - \frac{\mathrm{e}^{-i\pi t}}{2i}H(g\,\mathrm{e}^{i\pi\,\cdot})(t)\end{aligned}$$

for almost all $t \in \mathbb{R}$, where H denotes the Hilbert transform. Further, we see that

$$\begin{aligned}\|L_\pi g\|_p &\le \frac{1}{2}\|H(g\,\mathrm{e}^{-i\pi\,\cdot})\|_p + \frac{1}{2}\|H(g\,\mathrm{e}^{i\pi\,\cdot})\|_p \\ &\le \frac{1}{2}C_H(p)\|g\,\mathrm{e}^{-i\pi\,\cdot}\|_p + \frac{1}{2}C_H(p)\|g\,\mathrm{e}^{i\pi\,\cdot}\|_p \\ &= C_H(p)\|g\|_p, \end{aligned} \tag{34}$$

because $H\colon L^p(\mathbb{R}) \to L^p(\mathbb{R})$ is a bounded operator, satisfying $\|Hf\|_{L^p(\mathbb{R})} \le C_H(p)\|f\|_{L^p(\mathbb{R})}$ for all $f \in L^p(\mathbb{R})$, where

$$C_H(p) := \|H\| = \begin{cases}\tan\left(\frac{\pi}{2p}\right), & 1 < p \le 2, \\ \cot\left(\frac{\pi}{2p}\right), & 2 < p < \infty,\end{cases}$$

is a constant that depends only on p [44].

Let $1 < p < \infty$ be arbitrary but fixed and q such that $1/p+1/q = 1$. It can be shown that for $f \in \mathcal{B}_\pi^p$ and $g \in L^q(\mathbb{R})$, we have

$$\begin{aligned}\int_{-\infty}^{\infty} f(t)g(t)\,\mathrm{d}t &= \int_{-\infty}^{\infty}(L_\pi f)(t)g(t)\,\mathrm{d}t \\ &= \int_{-\infty}^{\infty} f(t)(L_\pi g)(t)\,\mathrm{d}t.\end{aligned}$$

Since $f \in \mathcal{B}_\pi^p$ and $L_\pi g \in \mathcal{B}_\pi^q$, we can use [17, p. 50, Th. 6.11] to obtain

$$\int_{-\infty}^{\infty} f(t)(L_\pi g)(t)\,\mathrm{d}t = \sum_{k=-\infty}^{\infty} f(k)(L_\pi g)(k).$$

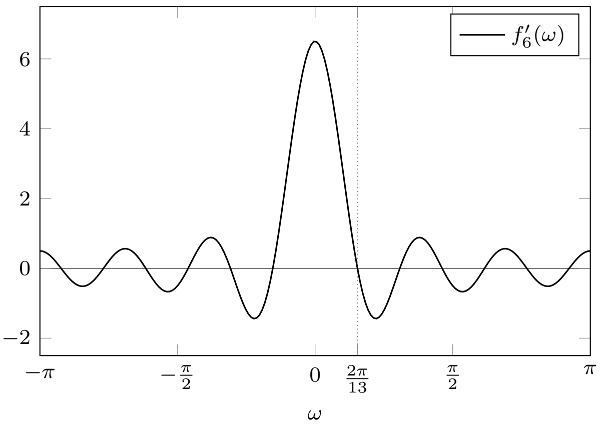

Fig. 4. Plot of the function f_6' for $N = 6$.

Hence, it follows that

$$\begin{aligned}\left|\int_{-\infty}^{\infty} f(t)g(t)\,\mathrm{d}t\right| &\le \sum_{k=-\infty}^{\infty}|f(k)(L_\pi g)(k)| \\ &\le \left(\sum_{k=-\infty}^{\infty}|f(k)|^p\right)^{\frac{1}{p}}\left(\sum_{k=-\infty}^{\infty}|(L_\pi g)(k)|^q\right)^{\frac{1}{q}} \\ &\le \left(\sum_{k=-\infty}^{\infty}|f(k)|^p\right)^{\frac{1}{p}}(1+\pi)\|L_\pi g\|_q \\ &\le \left(\sum_{k=-\infty}^{\infty}|f(k)|^p\right)^{\frac{1}{p}}(1+\pi)C_H(q)\|g\|_p, \end{aligned} \tag{35}$$

where we used Nikol'skiĭ's inequality [17, p. 49, Th. 6.8] in the second to last inequality and (34) in the last. Since (35) is true for all $g \in L^q(\mathbb{R})$, we obtain

$$\begin{aligned}\|f\|_p &= \sup_{\substack{g\in L^q(\mathbb{R}) \\ \|g\|_q\le 1}}\left|\int_{-\infty}^{\infty} f(t)g(t)\,\mathrm{d}t\right| \\ &\le \left(\sum_{k=-\infty}^{\infty}|f(k)|^p\right)^{\frac{1}{p}}(1+\pi)C_H(q)\end{aligned}$$

for all $f \in \mathcal{B}_\pi^p$.

Appendix B
Proof of Lemma 2

Proof of Lemma 2. Let $N \in \mathbb{N}$ be arbitrary but fixed. Since

$$\sum_{k=1}^{N}\frac{1}{k}\sin(k\omega) = \int_0^\omega\sum_{k=1}^{N}\cos(k\tau)\,\mathrm{d}\tau,$$

it follows that

$$\sum_{k=1}^{N}\frac{1}{k}\sin(k\omega) = \int_0^\omega\frac{\sin\left(\frac{2N+1}{2}\tau\right)}{2\sin\left(\frac{\tau}{2}\right)}\,\mathrm{d}\tau - \frac{\omega}{2}$$

for all $N \in \mathbb{N}$ and $\omega \in \mathbb{R}$. Let

$$f_N(\omega) = \int_0^\omega\frac{\sin\left(\frac{2N+1}{2}\tau\right)}{2\sin\left(\frac{\tau}{2}\right)}\,\mathrm{d}\tau, \quad \omega \in \mathbb{R}.$$

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The zeros of f_N' on the positive real axis are given by $\{2\pi n/(2N+1)\}_{n\in\mathbb{N}}$, and we have $f_N'(\omega) > 0$ for $t \in [0, 2\pi/(2N+1))$. The function f_6' is plotted in Fig. 4 for illustrative purposes. Due to the general behavior of f_N', we see that $f_N(\omega)$ attains its maximum at $\omega = 2\pi/(2N+1)$. We have

$$\begin{aligned} f_N\left(\frac{2\pi}{2N+1}\right) &= \int_0^{\frac{2\pi}{2N+1}} \frac{\sin\left(\frac{2N+1}{2}\tau\right)}{2\sin\left(\frac{\tau}{2}\right)} \,\mathrm{d}\tau \\ &= \int_0^{\frac{2\pi}{2N+1}} \frac{1}{2} + \sum_{k=1}^{N} \cos(k\tau) \,\mathrm{d}\tau \\ &\leq \int_0^{\frac{2\pi}{2N+1}} \frac{2N+1}{2} \,\mathrm{d}\tau \\ &= \pi. \end{aligned}$$

Thus, for $0 < \omega < \pi$ it follows that

$$\begin{aligned} \sum_{k=1}^{N} \frac{1}{k}\sin(k\omega) &= f_N(\omega) - \frac{\omega}{2} \\ &< f_N(\omega) \\ &\leq f_N\left(\frac{2\pi}{2N+1}\right) \\ &\leq \pi. \end{aligned}$$

Further, we have $\sum_{k=1}^{N} \sin(k\omega)/k > 0$ for $0 < \omega < \pi$ [32, p. 62, Theorem 9.4]. Since the sum $\sum_{k=1}^{N} \sin(k\omega)/k$ is zero for $\omega = 0$ and $\omega = \pi$, and an odd 2π-periodic function, we see that $|\sum_{k=1}^{N} \sin(k\omega)/k| < \pi$ for all $\omega \in \mathbb{R}$. □

Appendix C
Proof of Lemma 3

Proof of Lemma 3. Let $\delta \in (0, 1/2]$ be arbitrary but fixed, and let

$$F_N(\omega) = \sum_{k=1}^{N} \frac{1}{k}\cos(k\omega).$$

We distinguish two cases: $N \leq 1/\delta$ and $N > 1/\delta$. We start with the case $N \leq 1/\delta$. Then we have

$$\begin{aligned} |F_N(\omega)| &\leq \sum_{k=1}^{N} \frac{1}{k}|\cos(k\omega)| \\ &\leq \sum_{k=1}^{N} \frac{1}{k} \\ &< 1 + \int_1^N \frac{1}{x}\,\mathrm{d}x \\ &= 1 + \log(N) \\ &\leq 1 + \log\left(\frac{1}{\delta}\right) \end{aligned} \tag{36}$$

for all $\omega \in \mathbb{R}$. Next we treat the case $N > 1/\delta$. Let $N_\delta = \lfloor 1/\delta \rfloor$ be the largest natural number such that $N_\delta \leq 1/\delta$. Note that $N_\delta > 1/\delta - 1 \geq 1$. We have

$$F_N(\omega) = \sum_{k=1}^{N_\delta} \frac{1}{k}\cos(k\omega) + \sum_{k=N_\delta+1}^{N} \frac{1}{k}\cos(k\omega). \tag{37}$$

For the first sum in (37), we have according to (36) that

$$\left|\sum_{k=1}^{N_\delta} \frac{1}{k}\cos(k\omega)\right| < 1 + \log\left(\frac{1}{\delta}\right) \tag{38}$$

for all $\omega \in \mathbb{R}$. For the second sum in (37), we observe that

$$\sum_{k=N_\delta+1}^{N} \frac{1}{k}\cos(k\omega) = \sum_{k=N_\delta+1}^{N} \frac{1}{k}(c_k(\omega) - c_{k-1}(\omega)),$$

where $c_k(\omega) = \sum_{n=1}^{k} \cos(n\omega)$, and use summation by parts to obtain

$$\begin{aligned} \sum_{k=N_\delta+1}^{N} \frac{1}{k}\cos(k\omega) = \frac{1}{N}c_N(\omega) - \frac{1}{N_\delta+1}c_{N_\delta}(\omega) \\ + \sum_{k=N_\delta+1}^{N-1} \frac{1}{k(k+1)}c_k(\omega). \end{aligned} \tag{39}$$

Next, we need an upper bound for $|c_k(\omega)|$, $k \in \mathbb{N}$, $\omega \in [\delta, \pi]$. Using the identity for the Dirichlet kernel, we see that

$$c_k(\omega) = \sum_{n=1}^{k} \cos(n\omega) = \frac{\sin\left(\frac{2k+1}{2}\omega\right)}{2\sin\left(\frac{\omega}{2}\right)} - \frac{1}{2},$$

and consequently, that

$$|c_k(\omega)| \leq \frac{1}{2\sin\left(\frac{\omega}{2}\right)} + \frac{1}{2}.$$

Since $\omega \in [\delta, \pi]$, we have $\sin(\omega/2) \geq \omega/\pi \geq \delta/\pi$. It follows that

$$|c_k(\omega)| \leq \frac{\pi}{2\delta} + \frac{1}{2} \tag{40}$$

for all $k \in \mathbb{N}$ and all $\omega \in [\delta, \pi]$. Now we can upper bound the tree terms on the right hand side of (39). Since $N > 1/\delta$ and $N_\delta + 1 > 1/\delta$, and by using (40), we see that

$$\left|\frac{1}{N}c_N(\omega)\right| < \delta\left(\frac{\pi}{2\delta} + \frac{1}{2}\right) = \frac{\pi+\delta}{2} \leq \frac{\pi}{2} + \frac{1}{4} \tag{41}$$

and

$$\left|\frac{1}{N_\delta+1}c_N(\omega)\right| < \delta\left(\frac{\pi}{2\delta} + \frac{1}{2}\right) = \frac{\pi+\delta}{2} \leq \frac{\pi}{2} + \frac{1}{4} \tag{42}$$

for all $\omega \in [\delta, \pi]$. For the third term in (39), we obtain

$$\begin{aligned} &\left|\sum_{k=N_\delta+1}^{N-1} \frac{1}{k(k+1)}c_k(\omega)\right| \\ &\leq \sum_{k=N_\delta+1}^{N-1} \frac{1}{k(k+1)}|c_k(\omega)| \\ &\leq \left(\frac{\pi}{2\delta} + \frac{1}{2}\right) \sum_{k=N_\delta+1}^{N-1} \frac{1}{k(k+1)} \\ &< \left(\frac{\pi}{2\delta} + \frac{1}{2}\right) \sum_{k=N_\delta+1}^{\infty} \frac{1}{k^2} \\ &< \left(\frac{\pi}{2\delta} + \frac{1}{2}\right) \int_{N_\delta}^{\infty} \frac{1}{x^2}\,\mathrm{d}x \\ &= \left(\frac{\pi}{2\delta} + \frac{1}{2}\right) \frac{1}{N_\delta}, \end{aligned}$$

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where we used (40) in the second inequality. Since $\delta \leq 1/2$ and $N_\delta > 1/\delta - 1$, we have

$$\left(\frac{\pi}{2\delta} + \frac{1}{2}\right)\frac{1}{N_\delta} < \frac{\pi}{2(1-\delta)} + \frac{1}{2N_\delta} \leq \pi + \frac{1}{2},$$

and consequently

$$\left|\sum_{k=N_\delta+1}^{N-1} \frac{1}{k(k+1)} c_k(\omega)\right| < \pi + \frac{1}{2} \tag{43}$$

for all $\omega \in [\delta, \pi]$. Combining (37), (38), (39), (41), (42), and (43), we see that for $N > 1/\delta$ and all $\omega \in [\delta, \pi]$, we have

$$\begin{aligned} |F_N(\omega)| &< 1 + \log\left(\frac{1}{\delta}\right) + 2\left(\frac{\pi}{2} + \frac{1}{4}\right) + \pi + \frac{1}{2} \\ &= \log\left(\frac{1}{\delta}\right) + 2 + 2\pi. \end{aligned}$$ □

Appendix D
Computability of $\|g_N\|_p$

Let $N \in \mathbb{N}$ be arbitrary but fixed. Next, we will show that the norm $\|g_N\|_p$ is computable. Let

$$g_N^M(t) = \begin{cases} g_N(t), & |t| \leq M, \\ 0, & |t| > M. \end{cases}$$

It can be shown that g_N^M is a computable function on $[-M, M]$ and that

$$\|g_N^M\|_p = \left(\int_{-M}^{M} |g_N^M(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}}$$

is computable [9, p. 35, Theorem 5]. We have

$$\begin{aligned} \big|\|g_N\|_p - \|g_N^M\|_p\big| &\leq \|g_N - g_N^M\|_p \\ &= \left(\int_{|t|>M} |g_N(t)|^p \, \mathrm{d}t\right)^{\frac{1}{p}} \\ &\leq \sum_{k=1}^{N} \frac{1}{k\pi^2}\left(\int_{|t|>M} \frac{1}{|t-k|^{2p}}\right)^{\frac{1}{p}}. \end{aligned}$$

We treat the integral on the right hand side of the inequality next. For $M > N$ and $1 \leq k \leq N$, we obtain

$$\int_{|t|>M} \frac{1}{|t-k|^{2p}} \, \mathrm{d}t = \int_{-\infty}^{-M} \frac{1}{|t-k|^{2p}} \, \mathrm{d}t + \int_{M}^{\infty} \frac{1}{|t-k|^{2p}} \, \mathrm{d}t.$$

For the first integral we have

$$\begin{aligned} \int_{-\infty}^{-M} \frac{1}{|t-k|^{2p}} \, \mathrm{d}t &< \int_{-\infty}^{-M} \frac{1}{|t|^{2p}} \, \mathrm{d}t \\ &= \frac{1}{M^{2p-1}(2p-1)}, \end{aligned}$$

and for the second integral

$$\begin{aligned} \int_{M}^{\infty} \frac{1}{|t-k|^{2p}} \, \mathrm{d}t &< \int_{M}^{\infty} \frac{1}{|t-N|^{2p}} \, \mathrm{d}t \\ &= \frac{1}{(M-N)^{2p-1}(2p-1)}. \end{aligned}$$

Hence, it follows that

$$\begin{aligned} &\big|\|g_N\|_p - \|g_N^M\|_p\big| \\ &< \sum_{k=1}^{N} \frac{1}{k\pi^2}\left(\frac{1}{M^{2p-1}(2p-1)} + \frac{1}{(M-N)^{2p-1}(2p-1)}\right)^{\frac{1}{p}}, \end{aligned}$$

which shows that the computable sequence of computable numbers $\{\|g_N^M\|_p\}_{M\in\mathbb{N}}$ converges effectively to $\|g_N\|_p$. Hence, $\|g_N\|_p$ is computable.

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